Final Exam

CAS MA 511

Due date: The final exam is due at 5:00 PM EDT on Saturday July 3.

Instructions: Complete the following problems. You may consult your notes, the course notes, and the course textbook, but you may NOT utilize other sources or discuss problems with other students. Make sure that all of your work is fully justified and clearly explained.

1. Identify and fully explain the flaw in the following "proof" that all pens are the same color.

   Proof. We will prove this by induction. First, note the base case that a single pen is the same color as itself. Now, assume for the sake of induction that any set of n or fewer pens are the same color. We will now show that it follows that all sets of $n+1$ pens must be the same color. Let P be an arbitrary set of $n+1$ pens and $x, y \in P$ be arbitrary and distinct pens. The sets $P \setminus \{x\}$ and $P \setminus \{y\}$ each consist of n pens so by our inductive hypothesis, all pens in these sets have the same color. The set $(P \setminus \{x\}) \cap (P \setminus \{y\}) = P \setminus \{x, y\}$ has $n-1$ pens so, again these are all the same color by our hypothesis. The $n-1$ pens in this set are the same color as each other and they are each the same color as the pens x and y so the pens x and y have the same color. x and y were arbitrary pens in P so all pens in P are the same color. The set P was an arbitrary set of $n+1$ pens so all sets of $n+1$ pens are the the same color. There are only finitely many pens so by induction we can conclude that all pens are the same color. □

2. In this problem, we will explore a different notion of convergence. Consider the collection $\mathcal{O}$ of sets given by:

   $$\mathcal{O} = \{A \subset \mathbb{R} : A^c \text{ is finite}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$$

   (a) Prove that a countable union of sets in $\mathcal{O}$ is also in $\mathcal{O}$.

   (b) Prove that a finite intersection of sets in $\mathcal{O}$ is also in $\mathcal{O}$.

   (c) Given a sequence (x_n) of real numbers and a real number L, we will say that (x_n) $\mathcal{O}$-**converges** to L, if for every element $O \in \mathcal{O}$, we have that $L \in O$ implies that there exists an $N \in \mathbb{N}$ such that $x_n \in O$ for all $n \geq N$. Prove that the familiar sequence $(\frac{1}{n})$ $\mathcal{O}$-converges to 0.

   (d) Prove that the sequence $(\frac{1}{n})$ also $\mathcal{O}$-converges to 1. Furthermore, prove that the sequence $(\frac{1}{n})$ also $\mathcal{O}$-converges to any $L \in \mathbb{R}$.

   (e) Prove that the sequence $(n) = \{1, 2, 3, \dots\}$ $\mathcal{O}$-converges to any $L \in \mathbb{R}$.

3. We saw that if the $f'(x) < 0$ on some open interval (a, b), then $f(x)$ is decreasing on that interval. However, the same need not be true at a single point. Consider the function:

   $$f(x) = \begin{cases} -\frac{x}{2} + x^2 \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Prove that $f(x)$ is differentiable on $\mathbb{R}$ and that $f'(0) < 0$.

   (b) Prove that $f(x)$ is *not* decreasing on an open interval that contains 0.

4. We saw that monotone functions have countably many discontinuities. We also saw that Thomae's function t was discontinuous only on the rationals, i.e. $D_t = \mathbb{Q}$. However, t is not monotone on any interval. So, can we construct a function which is discontinuous on $\mathbb{Q}$ *and* monotone? Given an enumeration $\{q_1, q_2, q_3, \dots\}$ of $\mathbb{Q}$, we can define a step function for each q_n:

$$f_n(x) = \begin{cases} \frac{1}{3^n} & \text{if } x > q_n \\ 0 & \text{if } x \leq q_n \end{cases}$$

Then, we can consider $f(x) = \sum_{n=1}^{\infty} f_n(x)$.

(a) Prove that $f(x)$ is well-defined, in the sense that it always converges.
(b) Prove that $f(x)$ is a monotone function.
(c) Prove that $D_f = \mathbb{Q}$.

5. We saw that the Weierstrass function was no-where differentiable. In this problem, we will explore integrating the Weierstrass function. Consider the Weierstrass function

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{5^n} \cos(8^n \pi x)$$

(a) Prove that $f(x)$ is integrable.
(b) Find an antiderivative $F(x)$ of $f(x)$ such that $F(0) = 0$. (You may assume that if $\sum_{n=1}^{\infty} |\int f_n|$ converges, then $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.)
(c) What is the value of $\int_{-\frac{1}{2}}^{\frac{1}{2}} f$?

6. Recall that a **metric** on a set X is a map $d : X \times X \to [0, \infty)$ such that the following hold for all $x, y, z \in X$:

- $d(x, x) = 0$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Then, the pair $(X, d(x, y))$ is called a **metric space**. Recall that an **equivalence relation** $\sim$ on a set X is a binary relation such that the following hold for all $x, y, z \in X$:

- $x \sim x$
- If $x \sim y$, then $y \sim x$.
- If $x \sim y$ and $y \sim z$, then $x \sim z$.

Now, consider the metric space $(\mathbb{R}, |x-y|)$ and the equivalence relation given by $x \sim x'$ if $x - x' \in \mathbb{Z}$.

(a) Prove that $d([x], [y]) = \inf\{|x' - y'| : x' \in [x], y' \in [y]\}$ is a valid metric on the set of equivalence classes of this relation.
(b) What is the value of $d([\pi], [1000])$?

7. Recall that a metric space is **complete** if every Cauchy sequence in that space is convergent. Consider the metric space ℓ^∞ consisting of bounded sequences of real numbers with the metric $d_\infty((x_n), (y_n)) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}$.

(a) If $x_n = |\sin(n-1)|$ and $y_n = |\cos(n-1)|$, then what is the value of $d_\infty((x_n), (y_n))$?
(b) Write out the definition for a sequence in ℓ^∞ to be Cauchy. Note that sequences in ℓ^∞ are sequences of sequences of real numbers!
(c) Prove that $(\ell^\infty, d_\infty((x_n), (y_n)))$ is a complete metric space by showing that Cauchy sequences in ℓ^∞ converge.

BONUS. Pick a topic from the course that you found interesting and write a problem/solution that you would use to teach it to someone else.