Homework Solutions #2

CAS MA 511

Problem. 2.4.2

a) The argument presented in the problem shows that **if the limit does exist**, then its value must be $\frac{3}{2}$. It does not show that the limit exists and the limit, in fact, does not exist.

b) The argument can be applied after we first show the sequence converges. If we show it is converges, this approach gives a limit of $\frac{3+\sqrt{5}}{2}$. It is easy to show that y_n is increasing. If $y_{n+1} \geq y_n$, then $\frac{1}{y_{n+1}} \leq \frac{1}{y_n}$ and $3 - \frac{1}{y_{n+1}} \geq 3 - \frac{1}{y_n}$. Since y_n is increasing and $y_1 \geq 0$, $y_n \geq 0$. This means that $3 - \frac{1}{y_n} \leq 3$. y_n is monotone and bounded so it converges.

Problem. 2.4.4
The two statements of the Archimedean property are equivalent as we have seen so we will just show that $\mathbb{N}$ is not bounded in $\mathbb{R}$. Let $x_n = n$. This sequence is increasing so, if we assume for the sake of contradiction that $\mathbb{N}$ is bounded in $\mathbb{R}$, then the Monotone Convergence Theorem says that the sequence converges to some limit L which is an upper bound. If we take $\varepsilon = 1$, then the definition of convergence say that $L - 1 < n < L + 1$ when $n \geq N$ for some $N \in \mathbb{N}$. Considering $N + 1$ shows that $L < N + 1$, so L is not an upper bound. We have a contradiction so $\mathbb{N}$ must not be bounded in $\mathbb{R}$.

Problem. 2.5.2

a) True. The subsequence $y_n = x_{n+1}$ is a proper subsequence so it converges to a limit L. For $\varepsilon > 0$, let N be such that $N - 1$ satisfies the condition for convergence for y_n. This means that N is a valid choice for x_n.

b) True. A subsequence diverging means that for some $\varepsilon > 0$, there are points further than ε from any candidate limit which continue into the tail. These points are in the original so they disprove convergence for the same ε.

c) True. If the sequence x_n is bounded, it has a convergent subsequence, $a_n \to a$. Because x_n diverges, there is $\varepsilon > 0$ such that there are always points further from a than ε. There are infinitely many of these so consider them as a subsequence, y_n. This is bounded so it has a convergent subsequence. The terms are bounded away from a so the limit they converge to must be different.

d) True. The subsequence converges so it is bounded. Every term in the sequence is below a term in the subsequence so the sequence itself is bounded. The entire sequence is monotone and bounded so it converges.

Problem. 2.5.5
Assume for the sake of contradiction that the sequence a_n does not converge. Choose $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there is some $n > N$ with $|a_n - a| > \varepsilon$. There are infinitely many terms so we can define a subsequence, b_n, of these terms. This subsequence is bounded so it has a convergent subsequence c_n. c_n is a convergent subsequence of a_n but $|c_n - a| > \varepsilon$ for all n so $c_n \not\to a$. This is a contradiction so the sequence must converge.

Problem. 2.6.3

a) Because x_n and y_n are Cauchy sequences, for any $\varepsilon > 0$ we can choose $n_1, n_2 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \text{ if } n, m \geq N_1$$
$$|y_n - y_m| < \frac{\varepsilon}{2} \text{ if } n, m \geq N_2$$

Let $N = \max\{N_1, N_2\}$. If $n, m \geq N$, then the triangle inequality gives the desired result

$$\begin{aligned} |(x_n + y_n) - (x_m + y_m)| &= |x_n - x_m + y_n - y_m| \\ &\leq |x_n - x_m| + |y_n - y_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

b) Because x_n and y_n are Cauchy sequences, there exists M_1 and M_2 such that $|x_n| < M_1$ and $|y_n| < M_2$ and for any $\varepsilon > 0$ we can choose $n_1, n_2 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2M_2} \text{ if } n, m \geq N_1$$
$$|y_n - y_m| < \frac{\varepsilon}{2M_1} \text{ if } n, m \geq N_2$$

Let $N = \max\{N_1, N_2\}$. If $n, m \geq N$, then the triangle inequality gives the desired result

$$\begin{aligned} |(x_n y_n) - (x_m y_m)| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &\leq |x_n(y_n - y_m)| + |(x_n - x_m)y_m| \\ &\leq |x_n||y_n - y_m| + |x_n - x_m||y_m| \\ &< M_1 \frac{\varepsilon}{2M_1} + \frac{\varepsilon}{2M_2} M_2 = \varepsilon \end{aligned}$$

Problem. 2.7.1

a) a_n is decreasing and goes to 0 so $|s_n - s_{n+1}| = |a_{n+1}|$ is decreasing and going to 0. The triangle inequality shows that the distance between any pair of terms, s_n and s_{n+m} goes to 0. This means s_n is a Cauchy sequence so it must converge.

b) Consider the sequence of nested closed intervals $I_{2n-1} = [s_{2n}, s_{2n-1}]$ and $I_{2n} = [s_{2n}, s_{2n+1}]$. These are valid intervals because $s_{2n} = s_{2n-1} - a_{2n}$ and $s_{2n+1} = s_{2n} + a_{2n+1}$. Furthermore, we can show that they are nested because for each $n \in \mathbb{N}$ we compute $s_{2n-1} = s_{2n} + a_{2n} \geq s_{2n} + a_{2n+1} = s_{2n+1}$ and a similar result for even indexed intervals. The length of interval I_k is $|a_{k+1}|$ so the length of the intervals is going to 0. The intersection of all of these intervals is non-empty so there is an element, L inside of it. The choice of a sufficiently small interval shows all s_k are within ε of L for k sufficiently large.

c) $s_{2n} = s_{2n-2} + a_{2n-1} - a_{2n+2} \geq s_{2n-2} + a_{2n-1} - a_{2n+1} = s_{2n-2}$ so the even terms of s_k are increasing. A similar calculation shows that the odd terms are decreasing. We have also shown that $s_{2n} \leq s_{2n-1}$ in our previous work. This shows that both sequences converge. If one failed to converge, it must grow without bound, this would eventually be greater (or less) than all terms of the other sequence giving a contradiction. So, both are converging to their respective limits. The distance between the corresponding odd an even terms goes to 0 so each sequence can be made arbitrarily close to the other. This means there limits must be the same so the over all limit must exist.

Problem. 2.7.6

a) False. Consider $a_n = 1$. Clearly it is bounded but the sequence of partial sums is $s_n = n$.

b) True. any subsequence of a convergent sequence converges to the same limit so taking any subsequence of the partial sums gives subvergence.

c) True. The series of absolute values is increasing so if it subverges, then it must actually converge. If it converges, the series of normal terms converges as well so it must be subvergent.

d) False. Consider $a_n = (-1)^n n$. The sequence of partial sums is $s_n = (-1)^n$ which clearly subverges by taking alternating terms. a_n has terms which are always distance 2 or more from each other so no subsequences can be Cauchy and so cannot convergent.

Problem. 2.7.8

a) True. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $|a_n| \to 0$. We can choose $N \in \mathbb{N}$ such that $|a_n| < 1$ for $n \geq N$. This means that all $|a_n^2| < |a_n|$ for $n \geq N$. We then compute the following which shows the result

$$\begin{aligned}\sum_{n=1}^{\infty}|a_n^2| &= \sum_{n=1}^{N-1}|a_n^2| + \sum_{n=N}^{\infty}|a_n^2| \\ &< \sum_{n=1}^{N-1}|a_n^2| + \sum_{n=N}^{\infty}|a_n| \\ &< M + \sum_{n=1}^{\infty}|a_n| \\ &< M + S\end{aligned}$$

b) False. Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. $\sum_{n=1}^{\infty} a_n$ converges by alternating series test but $a_n b_n = \frac{1}{n}$ so $\sum_{n=1}^{\infty} a_n b_n$ diverges

c) True. The statement that $\sum_{n=1}^{\infty} a_n$ converges conditionally implies that it does **not** converge absolutely. Assume for the sake of contradiction that $\sum_{n=1}^{\infty} n^2 a_n$ converges. This means that $n^2 a_n \to 0$ so there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|n^2 a_n| < 1$. We conclude that $|a_n| < \frac{1}{n^2}$. By the comparision and p-tests, we can see that $\sum_{n=1}^{\infty}|a_n|$ converge, contradicting the statement that the series converges conditionally.

Problem. 2.8.7

a) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_i b_j| = \sum_{i=1}^{\infty}|a_i|\sum_{j=1}^{\infty}|b_j|$. $\sum_{j=1}^{\infty} b_j$ converges absolutely so letting $\sum_{j=1}^{\infty}|b_j| = \beta$ we see that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_i b_j| = \sum_{i=1}^{\infty}|a_i|\beta = \beta\sum_{i=1}^{\infty}|a_i|$. $\sum_{i=1}^{\infty} a_i$ converges absolutely so letting $\sum_{i=1}^{\infty}|a_i| = \alpha$ we see that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_i b_j| = \alpha\beta$. The iterated sum converges absolutely so we are able to apply theorem 2.8.1.

b) The doubly indexed array we need for theorem 2.8.1 is a_ib_i.

$$\begin{aligned}\left|\sum_{i=1}^n\sum_{j=1}^n a_ib_j - AB\right| &= \left|\left(\sum_{i=1}^n a_i\right)\left(\sum_{j=1}^n b_j\right) - AB\right| \\ &= \left|\left(\sum_{i=1}^n a_i\right)\left(\sum_{j=1}^n b_j\right) - A\left(\sum_{j=1}^n b_j\right) + A\left(\sum_{j=1}^n b_j\right) - AB\right| \\ &= \left|\left(\left(\sum_{i=1}^n a_i\right) - A\right)\left(\sum_{j=1}^n b_j\right) + A\left(\left(\sum_{j=1}^n b_j\right) - B\right)\right| \\ &\le \left|\sum_{i=1}^n a_i - A\right|\left|\sum_{j=1}^n b_j\right| + |A|\left|\sum_{j=1}^n b_j - B\right| \\ &\le \left|\sum_{i=1}^n a_i - A\right|\left(\sum_{j=1}^n |b_j|\right) + |A|\left|\sum_{j=1}^n b_j - B\right| \\ &\le \left|\sum_{i=1}^n a_i - A\right|\beta + |A|\left|\sum_{j=1}^n b_j - B\right|\end{aligned}$$

$\sum_{i=1}^\infty a_i = A$ so we can choose N_1 such that $|\sum_{i=1}^n a_i - A| < \frac{\varepsilon}{2\beta}$ for $n \geq N_2$ and any $\varepsilon > 0$. Similarly, we can choose $N_2 \in \mathbb{N}$ to make $\left|\sum_{j=1}^n b_j - B\right| < \frac{\varepsilon}{2A}$. Letting $N = \max\{N_1, N_2\}$, we can see that $\left|\sum_{i=1}^n\sum_{j=1}^n a_ib_j - AB\right| < \varepsilon$ for $n \geq N$, gioving us the result.

Problem. 3.2.2

a) The limit points of A are ± 1 which can be seen by taking the even and odd subsequences of the defining sequence. The limit points of B are $[0,1]$ because we can approximate any real by rationals.

b) A is not open because a small enough neighborhood around 2 is not contained in A. It is not closed because the limit point -1 is not in A. B is not open because any neighborhood around any rational must contain and irrational because they are dense. B is not closed because it does not contain the irrationals in $[0,1]$.

c) $A \setminus \{1\}$ are isolated points (take A and remove the limits points it contains). has no isolated points because all of its points are limit points.

d) The closure of A is $A \cup \{-1\}$ because -1 is the only limit point it does not already contain. The closure of B is $[0,1]$.

Problem. 3.2.3

a) $\mathbb{Q}$ is not open. Every point has an irrational in every neighborhood due to the density of irrationals. $\mathbb{Q}$ is not closed because we can approximate any irrational by a rational so they are all limit points.

b) $\mathbb{N}$ is not open. Every point has all neighborhood containing points not in the set. $\mathbb{N}$ is closed because all sequences of naturals which are not eventually constant, diverge so there are no limit points to not contain.

c) $\mathbb{R} \setminus \{0\}$ is open. We can show this by taking $\varepsilon < |x|$ gives the desired neighborhoods for any x. The set is not closed because $\frac{1}{n} \to 0$.

d) The set is not open because 1 has points not in the set in every neighborhood. The set is not closed because $\frac{\pi^2}{6}$ is the limit point of the sequence used to define the set.

e) The set is not open for the same reason as the previous set. The set is closed because the defining sequence does not converge so there are no limit points to not contain.

Problem. 3.3.6

a) We can prove finite sets have a max by induction. If sets of size n have a max, then we can find the max of sets of size $n+1$ by considering the set $\{\max\{A \setminus \{x_{n+1}\}, x_{n+1}\}$. This is a 2 element set which demonstrates that a size $n+1$ set has a max. Note: this induction argument requires that we use size 2 sets as a base case. To show this, we can just pick 1 element and if it is the max, we are done. If it is not the max, then the other element must be larger so that is clearly the max.
The statement is true for compact sets. These are closed and bounded so the sup exists and must be included so it is the max.
The statement is false for closed sets since $\mathbb{R}$ is closed and has no max.

b) This is true for finite sets because there are nm way of combining elements of size n and size m sets. This is the maximum cardinality of the sum of such sets.
This is true for compact sets. The sum of compact sets must also be bounded so we just need to show closure. But all sequences in the summed set are sums of sequences which are in the original sets. These sequences have convergent subsequences because they are necessarily bounded. The sum of these sub sequences must approach the same limit as the original sequences. However, this means that the limits of these subsequences exist and must sum to the limit of the original sequence. This shows closure.
This is false for closed sets. Consider the set of points in $\left\{n + \frac{1}{n} : n \in \mathbb{N}\right\} + \{-n : n \in \mathbb{N}\}$.

c) We can prove this for finite sets by contradiction. If we consider the sets $B_n = \bigcap_{i=1}^{n} A_n$, we get a set of finite and decreasing cardinality. There must be some $n \in \mathbb{N}$ such that $B_n = \emptyset$ for the infinite intersection to be $\emptyset$. But that contradicts the statement that all finite intersections are non-empty.
This is true for compact sets. We can define B_n the same as for the finite proof. All finite intersections are non-empty so we can define a sequence such that $x_n \in B_n$. This sequence is bounded so it has a convergent subsequence. This subsequence consists of elements which are in every A_n so the limit point is in every A_n. Therefore, the intersection is non-empty.
This is false for closed sets. The sets $[n, \infty)$ show this to be the case for closed sets.

Problem. 3.3.7

a) First, note that the statement is true for C_1. If $0 \leq s \leq \frac{2}{3}$, let $x_1, y_1 = \frac{s}{2}$. If $\frac{2}{3} \leq s \leq 1$, let $x_1 = \frac{1}{3}$ and $y_1 = 1 - x_n$. If $1 \leq s \leq \frac{4}{3}$, let $x_1 = 1$ and $y_1 = s - 1$. If $\frac{4}{3} \leq s \leq 2$, Let $x_1, y_1 = \frac{s}{2}$.
Now, assume that we have $x_n, y_n \in C_n$ such that $x_n + y_n = s$. If $x_n, y_n \in C_{n+1}$ then let $x_{n+1} = x_n$ and $y_{n+1} = y_n$.
If $x_n \in C_{n+1}$ but $y_n \notin C_{n+1}$, then we wish to offset x_n and y_n equally in opposite directions such that x_{n+1} is still in C_{n+1} and y_n is now in C_{n+1}. x_n is in a subinterval of C_{n+1} of length $\frac{1}{3^{n+1}}$. y_n is in a subinterval of C_n which is not in C_{n+1}. This subinterval is also length $\frac{1}{3^{n+1}}$ and is between 2 subintervals of C_{n+1}. We can define L_1 and L_2 as the distances from x_n to the lower and upper bound of the interval it is in. Similarly, we can define L_3 and L_4 as the relevant distances for y_n. If $L_1 \geq L_4$, then define $x_{n+1} = x_n - L_4$ and $y_{n+1} = y_n + L_4$. If

$L_1 < L_4$, then we compute the following

$$\begin{aligned}L_1 &< L_4\\ L_1 + L_2 &< L_2 + L_4\\ L_3 + L_4 &< L_2 + L_4\\ L_3 &< L_2\end{aligned}$$

In this case, we can define $x_{n+1} = x_n + L_3$ and $y_{n+1} = y_n - L_3$. If the roles of x_n and y_n are reversed, then we can use the same strategy
If $x_n, y_n \notin C_{n+1}$, then we conclude both are in intervals of length $\frac{1}{3^n}$ bounded by intervals in C_{n+1} and define L_k in a similar way. For similar reasons as above, we can determine that either $L_4 - L_1 < \frac{1}{3^{n+1}}$ or $L_2 - L_3 < \frac{1}{3^{n+1}}$ so we can define $x_{n+1} = x_n - \max\{L_1, L_4\}$ and $y_{n+1} = y_n + \max\{L_1, L_4\}$ or $x_{n+1} = x_n + \max\{L_2, L_3\}$ and $y_{n+1} = y_n - \max\{L_2, L_3\}$.

b) While x_n and y_n do not necessarily converge, they are bounded so they contain convergent subsequences. The limit points of these subsequences are elements of C which sum to s. We constructed this for arbitrary $s \in [0, 2]$ so we have shown that $[0, 2] \subseteq C + C$.