Homework Solutions #3

CAS MA 511

Problem (3.4.1)**.** If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Proof. Since P is closed and K is closed, $P \cap K$ is closed. Furthermore, since K is bounded and $P \cap K \subseteq K$, $P \cap K$ is bounded and hence compact. If P is nonempty and $p \in P$ and if $K = \{p\}$, then $P \cap K = \{p\}$ has an isolated point and hence is not perfect. So, $P \cap K$ must be closed, but is not necessarily perfect. □

Problem (3.4.4)**.** Repeat the Cantor construction from Section 3.1 starting with the interval $[0, 1]$. This time however, remove the open middle fourth from each component.

(a) Is the resulting set compact? Perfect?

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

Proof. For (a), let $\tilde{C}_0 = [0, 1]$ and let $\tilde{C}_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$ be the result of removing the open middle fourth of $\tilde{C}_0$. Continuing this way we obtain a sequence of sets $\tilde{C}_n$. Each $\tilde{C}_n$ is closed and hence the countable union

$$\tilde{C} = \bigcap_{n=1}^{\infty} \tilde{C}_n$$

is closed. Furthermore, we have $\tilde{C} \subseteq [0, 1]$, so $\tilde{C}$ is bounded and hence compact.

Now, let $x \in \tilde{C}$. Then, for $n \in \mathbb{N}$, since $x \in \tilde{C}$, we have $x \in \tilde{C}_n$ for each n. Since $x \in \tilde{C}_n$, which is the finite union of 2^{n-1} closed intervals of length $\frac{1}{4}\left(\frac{3}{8}\right)^{n-1}$, it must lie in one such interval. Let x_n be one of the endpoints of that interval such that $x_n \neq x$. Thus, we obtain a sequence (x_n) with $x_n \neq x$ such that $0 \leq |x_n - x| \leq \frac{1}{4}\left(\frac{3}{8}\right)^{n-1}$. Now, since $\frac{1}{4}\left(\frac{3}{8}\right)^{n-1} \to 0$, by the order limit theorem $|x_n - x| \to 0$ and hence $(x_n) \to x$. Thus, x is a limit point and since x was arbitrary this implies that $\tilde{C}$ has no isolated points and is therefore perfect.

For (b), at each step we remove 2^{n-1} open intervals whose length is $\frac{1}{4}\left(\frac{3}{8}\right)^{n-1}$, so the remaining length is:

$$1 - \frac{1}{4} - 2\frac{1}{4}\frac{3}{8} - 4\frac{1}{4}\left(\frac{3}{8}\right)^2 - \cdots = 1 - \sum_{n=1}^{\infty} 2^{n-1}\left(\frac{3}{8}\right)^{n-1}\frac{1}{4} = 1 - \frac{\frac{1}{4}}{1 - \frac{3}{4}} = 1 - 1 = 0$$

So, the length of $\tilde{C}$ is 0. Now, if we scale every real number by a factor of of $\frac{8}{3}$, then we get 2 copies of $\tilde{C}$, so the dimension of $\tilde{C}$ is $\frac{\log 2}{\log \frac{8}{3}} \approx 0.707$. □

Problem (3.5.2)**.** Replace each ______ with the word *finite* or *countable*, depending on which is more appropriate.

(a) The ______ union of F_σ sets is an F_σ set.

(b) The ______ intersection of F_σ sets is an F_σ set.

(c) The ______ union of G_δ sets is an G_δ set.

(d) The ______ intersection of G_δ sets is an G_δ set.

Proof. For (a), the countable union of F_σ sets is an F_σ set. Let $A_n = \bigcup_{m=1}^{\infty} A_{n,m}$ be an F_σ set. Then, we have:

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m} = \bigcup_{\{(n,m)\,:\,n,m\in\mathbb{N}\}} A_{n,m}$$

So, we just need to show that $\{(n,m) : n, m \in \mathbb{N}\}$ is countable. We can lay out the elements of this set as follows:

$$\begin{matrix} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & (2,3) & \cdots \\ (3,1) & (3,2) & (3,3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Now, just as we did for $\mathbb{Q}$ we can snake back and forth starting at the top left to form a bijection between this set and $\mathbb{N}$. The only difference is that we do not need to skip any pairs, since while $(1,1)$ and $(2,2)$ would correspond to the same fraction it need not be the case that $A_{1,1} = A_{2,2}$. Another way to state this fact is that $F_{\sigma\sigma} = F_\sigma$.

For (b), the finite intersection of F_σ sets is an F_σ set. If we show that the intersection of two F_σ sets is F_σ, then it will hold for any finite intersection by induction. If A and B are F_σ, then we have:

$$A \cap B = \left(\bigcup_{n=1}^{\infty} A_n\right) \cap \left(\bigcup_{m=1}^{\infty} B_m\right) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n \cap B_m$$

Thus, $A \cap B$ is indeed F_σ. If we took a countable intersection and tried to write it as a union it would require an uncountable index set, which is a bit tricky. Instead, we will show that the countable intersection of F_σ sets need not be F_σ using a proof by contradiction.

Suppose for contradiction that the countable intersection of F_σ sets is F_σ. Now, let A be a G_δ set, that is A is the countable intersection of open sets. Since open sets are F_σ, it follows that A is the countable intersection of F_σ sets and hence is F_σ by our induction. So, we have shown that any G_δ set is F_σ, however this is a contradiction since the set of irrationals $\mathbb{I}$ is a G_δ set, but not an F_σ set. Thus, the countable intersection of F_σ sets need not be F_σ.

For (c), the finite union of G_δ sets is an F_σ set. This follows from (a) and De Morgan's laws.

For (d), the countable intersection of G_δ sets is an F_σ set. This follows from (b) and De Morgan's laws. □

Problem (3.5.10)**.** Prove that the set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

Proof. Suppose for contradiction that $\mathbb{R}$ can be written as the countable union of nowhere-dense sets. That is, there are nowhere-dense sets $E_1, E_2, E_3, \ldots$ such that $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$. Then, for each E_n, we have $E_n \subseteq \overline{E_n} \subseteq \mathbb{R}$. Thus, we have:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E}_n \subseteq \mathbb{R}$$

However, this implies that $\mathbb{R} = \bigcup_{n=1}^{\infty} \overline{E}_n$. Then, taking complements, we have:

$$\emptyset = \mathbb{R}^c = \left(\bigcup_{n=1}^{\infty} \overline{E}_n \right)^c = \bigcap_{n=1}^{\infty} \overline{E}_n^c$$

Now, by exercise 3.5.8, each $\overline{E}_n^c$ is dense in $\mathbb{R}$. Thus, by theorem 3.5.2, the above intersection should be non-empty, so we have a contradiction, and we are forced to conclude that there are no such sets $E_1, E_2, E_3, \ldots$. □

Problem (4.2.6)**.** Decide if the following claims are true or false, and give short justifications for each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ sill also suffice.

(b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.

(c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3[f(x) - 2]^2 = 3(L-2)^2$.

(d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$, for any function g (with domain equal to the domain of f).

Proof. For (a), the claim is true. Suppose that given $\varepsilon > 0$, we have some $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $|f(x) - L| < \varepsilon$. Now consider $0 < \delta' < \delta$. Then, whenever we have $0 < |x - c| < \delta'$ it follows that $0 < |x - c| < \delta$ and hence $|f(x) - L| < \varepsilon$.

For (b), the claim is false. Let $f(x)$ be as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then, $\lim_{x\to 0} f(x) = 0$, but $f(0) = 1$.

For (c), the claim is true. If $\lim_{x\to a} f(x) = L$, then by the algebraic limit theorem for functional limits it follows that:

$$\begin{aligned} \lim_{x\to a} 3[f(x) - 2]^2 &= 3\left(\lim_{x\to a} f(x) - \lim_{x\to a} 2\right)\left(\lim_{x\to a} f(x) - \lim_{x\to a} 2\right) \\ &= 3(L-2)(L-2) \\ &= 3(L-2)^2 \end{aligned}$$

For (d), the claim is false. Let $f(x) = x$ on $(0,1)$ and let $g(x) = \frac{1}{x}$ on the same domain. Then, $\lim_{x\to 0} f(x) = 0$, but we have:

$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} 1 = 1 \neq 0$$

(Notice that g is unbounded! As the next exercise shows, this claim would be true if g were required to be bounded.) □

Problem (4.2.7)**.** Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on A. Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

Proof. Let g and f be real-valued functions on A with f bounded by $M > 0$. Then, assume that $\lim_{x\to c} g(x) = 0$. Let $\varepsilon > 0$. Now, since $\lim_{x\to c} g(x) = 0$ there exists a $\delta > 0$ such that for $0 < |x - c| < \delta$ we have $|g(x)| < \frac{\varepsilon}{M}$. Then, for $0 < |x - c| < \delta$ we have:

$$|g(x)f(x)| = |g(x)||f(x)| \leq |g(x)|M < \frac{\varepsilon}{M}M = \varepsilon$$

Thus, $\lim_{x\to c} g(x)f(x) = 0$. □

Problem (4.3.6)**.** Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x) + g(x)$ are continuous at 0.

(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.

(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

(d) A function $f(x)$ not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0.

(e) A function $f(x)$ not continuous at 0 such that $f(x)^3$ is continuous at 0.

Proof. For (a), let $f(x)$ be Dirichlet's function and let $g(x)$ be as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then, neither $f(x)$ nor $g(x)$ is continuous at 0, but $f(x)g(x) = 0$ and $f(x) + g(x) = 1$ are both constant and hence continuous at 0.

For (b), let $f(x)$ be continuous at 0, and let $g(x)$ not be continuous at 0. Now, suppose $f(x)+g(x)$ is continuous at 0. However, then since the difference of continuous functions is continuous, we have that $f(x) - (f(x) + g(x)) = g(x)$ is continuous at 0, which is a contradiction and hence the request is impossible.

For (c), we can let $f(x) = 0$ and let $g(x)$ be Dirichlet's function. Then, $f(x)$ is continuous everywhere and hence at 0, while $g(x)$ is nowhere continuous and hence not at 0. But, $f(x)g(x) = 0$, which is continuous at 0.

For (d), let $f(x)$ be as follows:

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$$

Then, $f(x)$ is not continuous at 0, but $f(x) + \frac{1}{f(x)} = \frac{5}{2}$ is constant and hence continuous at 0.

For (e), let $f(x)$ be not continuous at 0. Now, suppose that $f(x)^3$ is continuous at 0. Then, since $\sqrt[3]{x}$ is continuous on $\mathbb{R}$ and the composition of functions is continuous, we have that $\sqrt[3]{f(x)^3} = f(x)$ is continuous at 0, which is a contradiction and hence the request is impossible. □

Problem (4.3.13)**.** Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(b) Let $k = f(1)$, Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $x \in \mathbb{Z}$. Now prove that $f(r) = kr$ for any rational number r.

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Proof. For (a), if we let $x = y = 0$, then we have $f(0) = f(x + y) = f(x) + f(y) = f(0) + f(0)$. Subtracting $f(0)$ on both sides, we obtain $f(0) = 0$. Furthermore, if $y = -x$, then we have $0 = f(0) = f(x - x) = f(x) + f(-x)$ and hence $f(-x) = -f(x)$.

For (b), we first note that $f(n) = kn$ holds when $k = 1$. Now suppose that $f(n) = kn$ holds for n. Then, we have:

$$f(n + 1) = f(n) + f(1) = kn + k = (k + 1)n$$

Thus, the formula holds for $n+1$, whenever it holds for n, and since it holds for $n = 1$, by induction it holds for all $n \in \mathbb{N}$. Note that by (a), $f(0) = 0$ so that it also holds for 0. Now, consider $z \in \mathbb{Z}$ with $z < 0$.We must have $z = -n$, for some $n \in \mathbb{N}$ so that:

$$f(z) = f(-n) = -f(n) = -kn = zk$$

So, it holds for all $z \in \mathbb{Z}$. Finally, consider $r \in \mathbb{Q}$. Then $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ (and $q \neq 0$). So, we have:

$$kp = f(p) = f\left(q\frac{p}{q}\right) = qf\left(\frac{p}{q}\right) = qf(r)$$

Thus, diving by q, we obtain $f(r) = k\frac{p}{q} = kr$.

For (c), assume that f is continuous at 0. Then, consider some other point $c \in \mathbb{R}$, and let $\varepsilon > 0$. Then, since f is continuous at 0 we can find a $\delta > 0$ such that $|x| < \delta$ implies $|f(x)| < \varepsilon$. Now, consider $|x - c| < \delta$. This implies that $|f(x - c)| < \varepsilon$ and hence $|f(x) - f(c)| < \varepsilon$, by part (a) and our assumption that f is additive. Thus, f is continuous at c and hence continuous on $\mathbb{R}$.

Now, consider some irrational $i \in \mathbb{I}$, take a sequences of rationals $r_n \to i$. Then, since f is continuous on $\mathbb{R}$ we must have that $f(r_n) \to f(i)$. By part (c), $f(r_n) = kr_n \to ki$ and furthermore $f(i) = ki$. Thus, $f(x) = kx$ for all $x \in \mathbb{R}$. □

Problem (4.4.7)**.** Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon^2$ and consider x and y with $|x - y| < \delta$. Now, either $\sqrt{x} + \sqrt{y} < \varepsilon$ or $\sqrt{x} + \sqrt{y} \geq \varepsilon$. If $\sqrt{x} + \sqrt{y} < \varepsilon$, then, we have:

$$|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x}| + |\sqrt{y}| = \sqrt{x} + \sqrt{y} < \varepsilon$$

If on the other hand, $\sqrt{x} + \sqrt{y} \geq \varepsilon$, then we have:

$$|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \cdot \frac{|\sqrt{x} + \sqrt{y}|}{|\sqrt{x} + \sqrt{y}|} = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{\varepsilon} < \frac{\delta}{\varepsilon} = \varepsilon$$

So, either way $|\sqrt{x} - \sqrt{y}| < \varepsilon$, whenever $|x - y| < \delta$, and hence $\sqrt{x}$ is uniformly continuous. □

Problem (4.4.11)**.** Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

Proof. First, assume that g is continuous. Let O be open and let $x \in g^{-1}(O)$. Then, $g(x) \in O$ and since O is open there exists a $\varepsilon > 0$ such that $V_\varepsilon(g(x)) \subseteq O$. Now, since g is continuous, there exists $\delta > 0$ such that whenever $y \in V_\delta(x)$ it follows that $g(y) \in V_\varepsilon(g(x))$. In particular, this implies that $g(V_\delta(x)) \subseteq V_\varepsilon(g(x)) \subseteq O$ and hence $V_\delta(x) \subseteq g^{-1}(O)$. Thus, given $x \in g^{-1}(O)$, we can find an open neighborhood $V_\delta(x)$ in $g^{-1}(O)$ and hence $g^{-1}(O)$ is open.

Next, assume that whenever O is open it follows that $g^{-1}(O)$ is open. Now, let $\varepsilon > 0$. Then, since $V_\varepsilon(g(x))$ is open, $g^{-1}(V_\varepsilon(g(x)))$ is also open. Since $g^{-1}(V_\varepsilon(g(x)))$ is open and $x \in g^{-1}(V_\varepsilon(g(x)))$, there is some $V_\delta(x) \subset g^{-1}(V_\varepsilon(g(x)))$, which shows that g is continuous by the topological definition. □

Problem (4.5.2)**.** Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.

(b) A continuous function defined on a closed interval with range equal to an open interval.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.

(d) A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.

Proof. For (a), let f be a function on $(0, 1)$ defined by $f(x) = 0$. Then, the range $f((0, 1)) = [0, 0]$ is a closed interval.

For (b), the request is impossible since closed intervals are compact and the image of a compact set is compact and hence the range must be compact.

For (c), let f be a function on $(0, 1)$ defined by $f(x) = \left(\frac{1}{x}\right)\left(\frac{1}{1-x}\right)$. Then, the range of f is clearly unbounded. Furthermore, on $(0, 1)$, if $m(x) = \min\{x, 1 - x\}$, then $f(x) \geq \frac{1}{m(x)^2}$. In particular, the minimum value of $f(x)$ occurs when $x = 1 - x$, that is $x = \frac{1}{2}$. Since $\frac{1}{2} \in (0, 1)$, we have that the range $f((0, 1)) = [4, \infty)$ is closed.

For (d), the request is impossible since the image of connected sets must be connect and $\mathbb{R}$ is connected while $\mathbb{Q}$ is not. □

Problem (4.5.7)**.** Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

Proof. Let f be a continuous function on $[0, 1]$. Then, since f is continuous and $[0, 1]$ is compact and connected, we must have that $f([0, 1])$ is compact and connected. In particular, there must be $a, b \in \mathbb{R}$ such that $f([0, 1]) = [a, b]$. Since $f([0, 1]) \subseteq [0, 1]$, we also have that $a, b \in [0, 1]$.

Now, consider the function $g(x) = f(x) - x$ on $[0, 1]$. We have $g(0) = a$ and $g(1) = b - 1$. If either $a = 0$ or $b = 1$, then we have a fixed point. So, we just need to consider the case when $a > 0$ and $b < 1$. Then, the range of $g(x) = [b - 1, a]$ contains 0 in its interior. So, by the IVT there exists $c \in [0, 1]$ such that $g(c) = 0$. Finally, $g(c) = f(c) - c = 0$ implies that $f(c) = c$ so that f has a fixed point for this case as well. □

Problem (4.6.1)**.** Using modifications of Dirichlet's and Thomae's functions, construct a function $f : \mathbb{R} \to \mathbb{R}$ so that:

(a) $D_f = \mathbb{Z}^c$

(b) $D_f = \{x : 0 < x \leq 1\}$

Proof. For (a), consider the following function:

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

For any $x \in \mathbb{Z}^c$, we have $\sin(\pi x) \neq 0$. Furthermore, any rational sequence converging to x will converge to 0 and hence f is not continuous at x. Now, for $x \in \mathbb{Z}$, we have $\sin(\pi x) = 0$, and furthermore, since the modified Dirichlet function is continuous at 0, it follows that $f(x)$ is continuous at x.

For (b), consider the following function:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \text{ and } 0 \leq x \leq 1 \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x < 0 \text{ or } x > 1 \end{cases}$$

For $x < 0$ and $x > 1$, $f(x)$ is constant and hence continuous. On the interval $[0, 1]$, the function agrees with the modified Dirichlet function. This along with the fact that $f(x) = 0$ for $x \leq 0$ implies that $f(x)$ is continuous at 0. The left hand limit is clearly 0 and the right hand limit is 0 since the modified Dirichlet function is continuous at 0. □

Problem (4.6.2). Given a countable set $A = \{a_1, a_2, a_3, \dots\}$, define $f(a_n) = \frac{1}{n}$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

Proof. Suppose $x \in A$, then for $n \in \mathbb{N}$ consider $(x - \frac{1}{n}, x + \frac{1}{n})$. Since A has only countably many points and any interval in $\mathbb{R}$ has uncountably many points there must be some $x_n \in (x - \frac{1}{n}, x + \frac{1}{n})$ that is not in A. Now, since $\frac{1}{n} \to 0$, it follows that $x_n \to x$. Furthermore, since $x_n \notin A$ we have that $f(x_n) = 0$ for all n and hence $f(x_n) \to 0$. However, since $x \in A$, we have $f(x) \neq 0$ and hence f is not continuous at x.

Now, consider $x \notin A$ and consider an arbitrary sequence $(x_n) \to x$. If (x_n) contains finitely many points of A, then $f(x_n)$ will be eventually 0 and hence converge to 0. On the other hand, suppose that (x_n) contains infinitely many points of A. Let $\varepsilon > 0$ and let $N_1 \in \mathbb{N}$ be large enough such that $\varepsilon > \frac{1}{N_1}$. Now, suppose that n_i is the index of the i-th element of A as an element of (x_n) (assuming that $x_{n_i} \in A$). For example, if $a_3 \in (x_n)$, then there is some n_3 such that $x_{n_3} = a_3$. Then, consider:

$$N_2 = \max\{n_i : i < N_1\}$$

So, for $n \geq N_2$, if $x_n \in A$ it follows that the index of x_n in A is great than N_1 and hence $f(x_n) \leq \frac{1}{N_1} < \varepsilon$. If $x_n \notin A$, then $f(x) = 0 < \varepsilon$. So, either way $f(x_n) < \varepsilon$ and hence $f(x_n) \to 0$. Since x_n was arbitrary, it follows that $\lim_{y \to x} f(y) = 0 = f(x)$ and hence f is continuous at x. In summary, we have $D_f = A$, that is f is continuous on A^c. □