Homework Solutions #5

CAS MA 511

Problem (6.7.4)**.** First of all, we have:

$$\begin{aligned}
f(x) &= (1-x)^{\frac{1}{2}} \\
f'(x) &= -\frac{1}{2}(1-x)^{\frac{-1}{2}} \\
f''(x) &= -\frac{1}{2^2}(1-x)^{\frac{-3}{2}} \\
f^{(3)}(x) &= -\frac{1\cdot 3}{2^3}(1-x)^{\frac{-5}{2}} \\
&\vdots \\
f^{(n)}(x) &= -\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}(1-x)^{\frac{2n-1}{2}}
\end{aligned}$$

Thus, $a_0 = f(0) = 1$, and we have:

$$a_n = \frac{f^{(n)}}{n!} = -\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n} = -\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots 2n}$$

Problem (6.7.11)**.** Assume that f has continuous derivative on $[a,b]$. Then, by the Weierstrass approximation theorem, given $\varepsilon > 0$ there exists some polynomial $q(x)$ such that $|f'(x) - q(x)| < \min\{\varepsilon, \frac{\varepsilon}{|b-a|}\} \leq \varepsilon$ for all $x \in [a,b]$. Now, since $q(x)$ is a polynomial, we have:

$$q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

We would like to let $p(x)$ be an antiderivative of $q(x)$, so that $q(x) = p'(x)$. Now, we have not formally defined antiderivatives at this point, but we know what it should be for polynomials. Define $p(x)$ as follows:

$$p(x) = C + a_0 x + a_1\frac{x^2}{2} + \cdots a_n\frac{x^{n+1}}{n+1}$$

Then, $p'(x) = q(x)$. Note that we have free choice of C in our definition of $p(x)$, and we will choose it so that $f(a) - p(a) = 0$. Now, to relate $f(x) - p(x)$ and $f'(x) - p'(x)$, we can use the MVT. Let $x \in [a,b]$. Then, since $f(x) - p(x)$ is differentiable on $[a,x]$ there is some $c \in [a,x]$ such that:

$$\begin{aligned}
\frac{f(x) - p(x) - (f(a) - p(a))}{x-a} &= f'(c) - p'(c) \\
|f(x) - p(x)| &= |f'(c) - p'(c)||x-a| \\
&\leq |f'(c) - p'(c)||b-a| \\
&< |b-a|\frac{\varepsilon}{|b-a|} = \varepsilon
\end{aligned}$$

Thus, we have:

$$|f(x) - p(x)| < \varepsilon \quad \text{and} \quad |f'(x) - p'(x)| < \varepsilon$$

Problem (7.2.2)**.** Let $f(x) = \frac{1}{x}$ on $[1, 4]$ and let $P = \{1, \frac{3}{2}, 2, 4\}$. For (a), we have:

$$\begin{aligned} U(f,P) &= (\tfrac{3}{2} - 1) + \tfrac{2}{3}(2 - \tfrac{3}{2}) + \tfrac{1}{2}(4-2) = \tfrac{1}{2} + \tfrac{1}{3} + 1 = \tfrac{11}{6} \\ L(f,P) &= \tfrac{2}{3}(\tfrac{3}{2} - 1) + \tfrac{1}{2}(2 - \tfrac{3}{2}) + \tfrac{1}{4}(4-2) = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{13}{12} \\ U(f,P) - L(f,P) &= \tfrac{11}{6} - \tfrac{13}{12} = \tfrac{3}{4} \end{aligned}$$

For (b), if we add the point 3 to the partition to P. Then, we have:

$$\begin{aligned} U(f,P) &= (\tfrac{3}{2} - 1) + \tfrac{2}{3}(2 - \tfrac{3}{2}) + \tfrac{1}{2}(3-2) + \tfrac{1}{3}(3-2) + = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{3} \\ L(f,P) &= \tfrac{2}{3}(\tfrac{3}{2} - 1) + \tfrac{1}{2}(2 - \tfrac{3}{2}) + \tfrac{1}{3}(3-2) + \tfrac{1}{4}(4-3) = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{7}{6} \\ U(f,P) - L(f,P) &= \tfrac{5}{3} - \tfrac{7}{6} = \tfrac{1}{2} \end{aligned}$$

For (c), if we use the partition $P' = \{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$, then we have:

$$\begin{aligned} U(f,P) &= \tfrac{223}{140} \\ L(f,P) &= \tfrac{341}{280} \\ U(f,P) - L(f,P) &= \tfrac{3}{8} < \tfrac{2}{5} \end{aligned}$$

Problem (7.2.3)**.** For (a), let f be a bounded function. First, suppose that f integrable on $[a, b]$. Then, since f is integrable for each n there is a partition P_n such that:

$$U(f, P_n) - L(f, P_n) < \frac{1}{n}$$

Thus, we can obtain a sequence of partitions (P_n) such that $(U(f, P_n) - L(f, P_n)) \to 0$. On the other hand if we have such sequence of partitions (P_n), then given $\varepsilon > 0$, there exists a natural number N with $|U(f, P_N) - L(f, P_N)| < \varepsilon$. Thus, letting $P_\varepsilon = P_N$, we have the f is integrable by the integrability criterion. Furthermore, for $\varepsilon > 0$ there exists an N such that for $n \geq N$ we have:

$$\begin{aligned} U(f, P_n) - U(f) < \varepsilon + L(f, P_n) - U(f) \leq \varepsilon + L(f) - U(f) = \varepsilon \\ L(f) - L(f, P_n) < L(f) - U(f, P_n) + \varepsilon \leq L(f) - U(f) + \varepsilon = \varepsilon \end{aligned}$$

and hence

$$\lim_{n\to\infty} U(f, P_n) = U(f) \quad \text{and} \quad \lim_{n\to\infty} L(f, P_n) = L(f)$$

Thus, we have $\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$.

For (b), since $f(x) = x$ is increasing, on a given interval the supremum will be at the right endpoint and the infimum will be at the left endpoint. Thus, we have:

$$\begin{aligned} U(x, P_n) &= \sum_{k=1}^{n} \frac{1}{n}\frac{k}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{n+1}{2n} \\ L(x, P_n) &= \sum_{k=1}^{n} \frac{1}{n}\frac{k-1}{n} = \frac{1}{n^2}\sum_{k=1}^{n}(k-1) = \frac{1}{n^2}\sum_{k=0}^{n-1} k = \frac{1}{n^2}\frac{n(n-1)}{2} = \frac{n-1}{2n} \end{aligned}$$

For (c), we have:

$$\lim_{n\to\infty} U(x, P_n) - L(x, P_n) = \lim_{n\to\infty} \frac{n+1}{2n} - \frac{n-1}{2n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus, $f(x) = x$ is integrable on $[0, 1]$ and we have:

$$\int_0^1 x = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{2} = \frac{1}{2}$$

Problem (7.3.2)**.** For (a), suppose P is an arbitrary partition of $[0,1]$. Since the irrationals are dense in $\mathbb{R}$, every interval contains some irrational point and hence a point where Thomae's function t is zero and hence $L(t,P)=0$.

For (b), let $\varepsilon>0$. Then, there is some $N\in\mathbb{N}$ such that $\frac{1}{N+1}<\frac{\varepsilon}{2}\leq\frac{1}{N}$. Thus, the elements of $D_{\frac{\varepsilon}{2}}$ are of the form $\frac{p}{q}$ where $p\leq N$ and, since $\frac{p}{q}<1$, $p\leq q$. In particular, the possible elements can be written out:

$$\begin{array}{ccccc} \frac{1}{1} & & & & \\ \frac{1}{2} & \frac{2}{2} & & & \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & & \\ \vdots & & & \ddots & \\ \frac{1}{N} & \frac{2}{N} & \frac{3}{N} & \cdots & \frac{N}{N} \end{array}$$

This, list certainly has redundancies, but from it we can see that $D_{\frac{\varepsilon}{2}}$ has at most $1+2+3+\cdots+N=\frac{N(N+1)}{2}$ elements. In particular, it is finite!

For (c), we can choose P_ε so that there are very small subintervals around each of the finitely many points in $D_{\frac{\varepsilon}{2}}$. Choose these subintervals to each contain one point of $D_{\frac{\varepsilon}{2}}$ and have combined length less than $\frac{\varepsilon}{2}$. Their contribution to $U(f,P_\varepsilon)$ will be less than $\frac{\varepsilon}{2}$ since $t(x)\leq 1$. Now, for the remaining subintervals of P_ε, we know that $t(x)<\frac{\varepsilon}{2}$ and since their combined length will be less than 1, we have that $U(t,P_\varepsilon)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.

From (a) and (c), we have that $U(t,P_\varepsilon)-L(t,P_\varepsilon)<\varepsilon$ so that by the integrability criterion $t(x)$ is integrable on $[0,1]$ and since $L(t,P)=0$ we see that $\int_0^1 t=0$.

Problem (7.3.7)**.** For (a), suppose that $f:[a,b]\to\mathbb{R}$ is integrable and g satisfies $g(x)=f(x)$ for all but a finite number of points in $[a,b]$. If we can prove that g is integrable when it differs at only a single point, then we can show it by induction for any finite number of points. Suppose that $g(x)\neq f(x)$ only at a single point $c\in[a,b]$ and let M be such that $0\leq|g(c)-f(c)|<M$, Now, since f is integrable, for $\varepsilon>0$, we can find a partition P_ε such that:

$$U(f,P_\varepsilon)-L(f,P_\varepsilon)<\frac{\varepsilon}{2}$$

Now, we can refine P_ε by adding in points p_1 and p_2 such that $p_1,p_2\in(c-\frac{\varepsilon}{4M},c+\frac{\varepsilon}{4M})$ and there are no points of P_ε in between p_1 and p_2. (Note that one or both of p_1 or p_2 could already be in P_ε.) Call this new partition P'_ε and note that $P_\varepsilon\subseteq P'_\varepsilon$. Moreover, this new subinterval has length less than $\frac{\varepsilon}{2M}$ and the most it could change the increase the supremum or decrease the infimum is M. In particular, we have:

$$U(g,P'_\varepsilon)-L(g,P'_\varepsilon)\leq U(f,P'_\varepsilon)-L(f,P'_\varepsilon)+M\frac{\varepsilon}{2M}\leq U(f,P_\varepsilon)-L(f,P_\varepsilon)+\frac{\varepsilon}{2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

where the second $\leq$ follows since P'_ε is a refinement of P_ε. Thus, g is integrable by the integrability criterion.

For (b), Dirichlet's function differs from the zero function at the countably many points $\mathbb{Q}\cap[0,1]$, but is not integrable.

Problem (7.4.2)**.** For (a), note that $g(x)=-g(-x)$ and hence $\int_0^a g=-\int_{-a}^0 g$. Thus, we have:

(i) $\int_0^{-1} g+\int_0^1 g=-\int_{-1}^0 g+\int_0^1 g=\int_0^1 g+\int_0^1 g=2\int_0^1 g>0$

(ii) $\int_1^0 g+\int_0^1 g=-\int_0^1 g+\int_0^1 g=0$

(iii) $\int_1^{-2} g+\int_0^1 g=-\int_{-2}^1 g+\int_0^1 g=-\int_{-2}^0 g-\int_0^1 g+\int_0^1 g=-\int_{-2}^0 g=\int_0^2 g>0$

For (b), since f is integrable on $[b, c]$ and $a \in [b, c]$, we have:

$$-\int_c^b f = \int_b^c f = \int_b^a f + \int_a^c = -\int_a^b f + \int_a^c$$

Thus, adding the integrals with negative signs to both sides, we have:

$$\int_a^b f = \int_a^c + \int_c^b f$$

Problem (7.4.6). For (a), suppose that $f(x) \leq M$ on $[a, b]$. Then, we have:

$$|(f(x))^2 - (f(y))^2| = |f(x)+f(y)||f(x)-f(y)| \leq (|f(x)|+|f(y)|)|f(x)-f(y)| \leq 2M|f(x)-f(y)|$$

For (b), since f is integrable for $\varepsilon > 0$ there is a partition P_ε such that:

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \frac{\varepsilon}{2M}$$

Then, we have:

$$\begin{aligned}
U(f^2, P_\varepsilon) - L(f^2, P_\varepsilon) &= |U(f^2, P_\varepsilon) - L(f^2, P_\varepsilon)| \\
&= \sum_{k=1}^{n} |f^2(z_k) - f^2(y_k)|\Delta x_k \\
&\leq \sum_{k=1}^{n} 2M|f(z_k) - f(y_k)|\Delta x_k \\
&= 2M \sum_{k=1}^{n} |f(z_k) - f(y_k)|\Delta x_k \\
&= |U(f, P_\varepsilon) - L(f, P_\varepsilon)| \\
&< 2M\frac{\varepsilon}{2M} = \varepsilon
\end{aligned}$$

where z_k and y_k are the points in $[x_{k-1}, x_k]$ where f attains its minimum and maximum. Note that we need the absolute value signs because we do not know the signs of $f(z_k)$ and $f(y_k)$ and it could be the case that $f(y_k)$ is the supremum f and $f(z_k)$ is the infimum f.

For (c), suppose that f and g are integrable. Then $f + g$ is integrable, and by (b) we have that f^2, g^2 and $(f + g)^2$ are integrable. Thus, we have that

$$fg = \frac{1}{2}((f + g)^2 - f^2 - g^2))$$

is integrable.

Problem (7.5.6). For (a), since $h(x)$ and $k(x)$ are differentiable, we have:

$$(h \cdot k)'(x) = h(x)k'(x) + h'(x)k(x)$$

Since $h'(x)$ and $k'(x)$ are continuous the above function is integrable, and hence by the fundamental theorem of calculus, we have:

$$h(b)k(b) - h(a)k(a) = (h \cdot k)(b) - (h \cdot k)(a) = \int_a^b (h \cdot k)'(x)dx = \int_a^b h(x)k'(x)dx + \int_a^b h'(x)k(x)dx$$

Thus, subtracting the final integral from both sides, we have:

$$\int_a^b h(x)k'(x)dx = h(b)k(b) - h(a)k(a) - \int_a^b h'(x)k(x)dx$$

For (b), we only need that the derivatives are integrable, because then we can use exercise 7.4.6 to show that the products $h(x)k'(x)$ and $h'(x)k(x)$ and hence $(h \cdot k)'(x)$ are integrable.