# Lecture #1

MA 511, Introduction to Analysis

May 24, 2021

# The Irrationality of $\sqrt{2}$

- G.H. Hardy argues that mathematics "must be justified as art if it is to be justified at all."

**Theorem**

*There is no rational number whose square is 2.*

- What does this tell us about our conception of numbers?

$$\mathbb{N} = \{1, 2, 3, \dots\}$$
$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$$
$$\mathbb{Q} = \{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with } q \neq 0\}$$
$$\mathbb{R} = ???$$

- How well can we approximate $\sqrt{2}$ with a rational number? How many "numbers" is $\mathbb{Q}$ "missing?"

# Sets

- To write proofs and develop the theory of analysis, we need some agreed upon terminology and notation!

**Definition**

Intuitively, a **set** is any collection of objects, which are called the **elements** of the set. We write $x \in A$ if $x$ is an element of $A$. Given two sets $A$ and $B$, the **union** $A \cup B$ is defined by:

$$x \in A \cup B \text{ if } x \in A \text{ or } x \in B \text{ or both}$$

The **intersection** $A \cap B$ is defined by:

$$x \in A \cap B \text{ if } x \in A \text{ and } x \in B$$

*Example:* Natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$, "set builder notation" $D = \{x \in \mathbb{N} : x < 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, empty set $\varnothing$.

# Sets (cont.)

- How can we relate sets to each other?

**Definition**

$A$ is a **subset** of $B$, or $B$ contains $A$, written $A \subseteq B$, or $B \supseteq A$, if every element of $A$ is an element of $B$. This is called the **inclusion** relationship, and we have:

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

Most of the sets we will consider will be subsets of the real numbers $\mathbb{R}$. For $A \subseteq \mathbb{R}$ the **complement** of $A$, written $A^c$, is defined by:

$$A^c = \{x \in \mathbb{R} : x \notin A\}$$

*Example:* De Morgan's Laws:

$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c$$

# Functions

## Definition

Given two sets $A$ and $B$ a **function** $f$ from $A$ to $B$ is a rule or mapping that takes each element $x \in A$ and associates it with a single element $f(x)$ of $B$. In this case, we write $f : A \to B$ and call $A$ the **domain** of $f$. The **range** of $f$ is not necessarily all of $B$, but refers to the subset:

$$\{y \in B : f(x) = y \text{ for some } x \in A\}$$

*Example:* $f(x) = x^2 + x + 1$, Dirichlet function, absolute value.

## Proposition

*The absolute value function satisfies:*

$$|ab| = |a||b|$$

*and the* ***triangle inequality****:*

$$|a + b| \leq |a| + |b|$$

# Logic and Proofs

- **Direct proof ($P \Rightarrow Q$):** assume the hypothesis and proceed by rigorously logical deductions to demonstrate the conclusion.
  - *Example:* Proof of De Morgan's Laws

- **Proof by contradiction ($P$ and not $Q \Rightarrow$ contradiction):** assume the hypothesis, negate the conclusion and proceed by rigorously logical deductions until a contradiction arises.
  - *Example:* Proof of the irrationality of $\sqrt{2}$

- **Contrapositive proof (not $Q \Rightarrow$ not $P$):** negate the conclusion and proceed by rigorously logical deductions to demonstrate the negation of the hypothesis.

# Induction

- **Proof by induction:** For a proposition $P(n)$, depending on a natural number $n$, show that $P(1)$ holds and that if $P(n)$ holds, then $P(n+1)$ holds. By the following proposition, this implies that $P(n)$ holds for all $n \in \mathbb{N}$.

## Proposition

*If $S \subseteq \mathbb{N}$ with the property that:*

i. $1 \in S$
ii. *If $n \in S$, then $n+1 \in S$.*

*then $S = \mathbb{N}$.*
*(We may sometimes replace $\mathbb{N}$ with $\mathbb{N} \cup \{0\}$ and $1 \in S$ with $0 \in S$.)*

*Example:* Proof that:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$