Lecture #10

MA 511, Introduction to Analysis

June 8, 2021

Dirichlet's function

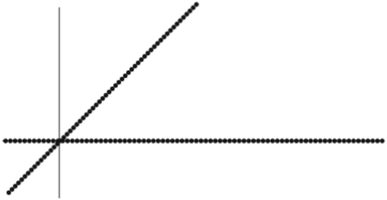
- Above is the graph of **Dirichlet's function** $g : \mathbb{R} \to \mathbb{R}$, which is given by:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- We will say that f is continuous at c if $\lim_{x \to c} f(x) = f(c)$, but how should we define functional limits $\lim_{x \to c} f(x)$? Consider sequences $f(x_n)$ where either $x_n \in \mathbb{Q}$ for all n or $x_n \notin \mathbb{Q}$ for all n.
- Dirichlet's function is a **nowhere-continuous** function on $\mathbb{R}$.

Modified Dirichlet function

- We want $\lim_{x\to c} f(x) = L$ to imply that $f(x_n) \to L$ for all sequences $(x_n) \to c$.

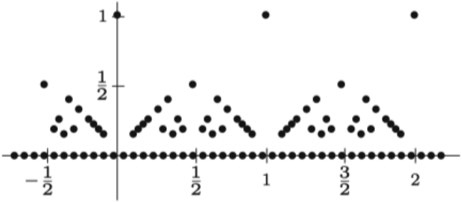


- Above is the graph of the **modified Dirichlet function** $h : \mathbb{R} \to \mathbb{R}$, which is given by:

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Now, the modified Dirichlet function is continuous at $x = 0$, but only at $x = 0$.

Thomae's function



- Above is the graph of **Thomae's function** $t : \mathbb{R} \to \mathbb{R}$, which is given by:

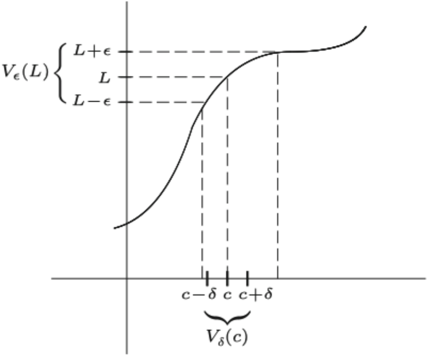
$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Thomae's function fails to be continuous at any rational point $x \in \mathbb{Q}$, however it is continuous at every irrational point $x \in \mathbb{I}$.
- Could a function be continuous on $\mathbb{Q}$, but fail to be continuous on $\mathbb{I}$?

Functional limits

Definition (Functional limit: ε–δ version)

Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = L$ provided that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \varepsilon$.



Functional limits (cont.)

Definition (Functional limit: topological version)

Let c be a limit point of the domain of $f : A \to \mathbb{R}$. We say $\lim_{x\to c} f(x) = L$ provided that, for every ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\varepsilon(L)$.

Definition (Infinite limits)

Let c be a limit point of the domain of $f : A \to \mathbb{R}$. Then, we say $\lim_{x\to c} f(x) = \infty$ if for all $M > 0$, we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

Definition (Limits at infinity)

We say $\lim_{x\to\infty} f(x) = L$ if for any $\varepsilon > 0$ we can find $M > 0$ such that whenever $x > M$, it follows that $|f(x) - L| < \varepsilon$.

Sequential criterion for functional limits

Theorem (Sequential criterion for functional limits)

Given a function $f : A \to \mathbb{R}$ and a limit point $c \in A$, the following statements are equivalent:

- i. $\lim_{x \to c} f(x) = L$
- ii. *For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.*

Corollary (Algebraic limit theorem for functional limits)

Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A.

- i. $\lim_{x \to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$
- ii. $\lim_{x \to c}[f(x) + g(x)] = L + M$
- iii. $\lim_{x \to c} f(x)g(x) = LM$
- iv. $\lim_{x \to c} f(x)/g(x) = L/M$ *provided* $M \neq 0$

Divergence criterion for functional limits

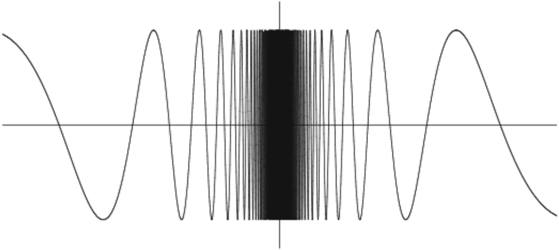
Corollary (Divergence criterion for functional limits)

Let f be a function defined on A, and let c be a limit point of A. If there exists two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and

$$\lim x_n = \lim y_n = c \text{ but } \lim f(x_n) \neq \lim f(y_n)$$

then we can conclude that the functional limit $\lim_{x \to c} f(x)$ does not exist.

- Consider the function $\sin\left(\frac{1}{x}\right)$ near $x = 0$:



- We want to make rigorous the intuitive idea of an "unbroken curve" without "jumps" or "holes."

Definition

A function $f : A \to \mathbb{R}$ is **continuous at a point** $c \in A$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \varepsilon$. If f is continuous for every point in the domain A, then we say that f **is continuous on** A.

- It is tempting to make the definition of continuous be that f is continuous at x if the following holds:

$$\lim_{x \to c} f(x) = f(c)$$

  However, what happens if c is an isolated point (i.e. c is not a limit point)?

Characterizations of continuity

Theorem (Characterizations of continuity)

Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:

- i *For all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \varepsilon$.*
- ii *For all $V_\varepsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implied $f(x) \in V_\varepsilon(f(c))$.*
- iii *If $(x_n) \to c$ (with $x_n \in A$), then $f(x_n) \to f(c)$.*

If c is a limit point of A, then the above conditions are equivalent to:

- iv $\lim_{x \to c} f(x) = f(c)$.

- Part iii is typically most useful for demonstrating that a function is *not* continuous.

Properties of continuous functions

Corollary (Criterion for discontinuity)

Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.

- We can leverage the algebraic limit theorem for functional limits to prove the following theorem, which allows us to build new continuous functions from ones we already know to be continuous.

Theorem (Algebraic continuity theorem)

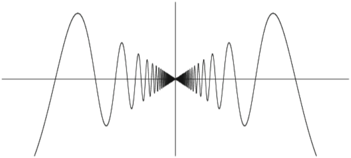
Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at a point $c \in A$.

i. $kf(x)$ *is continuous at* c *for all* $k \in \mathbb{R}$.
ii. $f(x) + g(x)$ *is continuous at* c.
iii. $f(x)g(x)$ *is continuous at* c.
iv. $f(x)/g(x)$ *is continuous at* c, *provided* $g(c) \neq 0$.

Properties of continuous functions (cont.)

- Is the following function continuous at $x = 0$?

$$g(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



- Polynomials, rational functions and square roots are all continuous, but what about $f(x) = \sqrt{x+1}$?

Theorem (Composition of continuous functions)

Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.