# Lecture #12

MA 511, Introduction to Analysis

June 10, 2021

# Sets of discontinuity

**Definition**

Given a function $f : \mathbb{R} \to \mathbb{R}$ the **set of discontinuities of** $f$, denoted by $D_f$, is the set of points where $f$ fails to be continuous.

- For Dirichlet's function $g(x)$, we saw that $D_g = \mathbb{R}$. For the modified Dirichlet function $h(x)$, we saw that $D_h = \mathbb{R} \setminus \{0\}$. For Thomae's function $t(x)$, we saw that $D_t = \mathbb{Q}$.
- For an arbitrary subset $A \subseteq \mathbb{R}$, can we find a function $f$ such that $D_f = A$?
- First, we will consider the possible $D_f$ for a simpler class of functions: monotone functions.

**Definition**

A function $f : A \to \mathbb{R}$ is **increasing on** $A$ if $f(x) \leq f(y)$ whenever $x < y$ and **decreasing** if $f(x) \geq f(y)$ whenever $x < y$ in $A$. A function is **monotone** if it is either increasing or decreasing.

# Right-hand and left-hand limits

- We need to formalize what we mean when we say that a function approaches a limit "from the right" or "from the left".

**Definition (Right-hand limit)**

Given a limit point $c$ of a set $A$ and a function $f : A \to \mathbb{R}$, we write $\lim_{x \to c^+} f(x) = L$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < x - c < \delta$. Equivalently, $\lim_{x \to c^+} f(x) = L$ if $\lim f(x_n) = L$ for all sequences $(x_n)$ such that $x_n > c$ and $\lim(x_n) = c$.

**Definition (Left-hand limit)**

Given a limit point $c$ of a set $A$ and a function $f : A \to \mathbb{R}$, we write $\lim_{x \to c^-} f(x) = L$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $-\delta < x - c < 0$. Equivalently, $\lim_{x \to c^-} f(x) = L$ if $\lim f(x_n) = L$ for all sequences $(x_n)$ such that $x_n < c$ and $\lim(x_n) = c$.

# Types of discontinuities

## Theorem

*Given $f : A \to \mathbb{R}$ and a limit point $c$ of $A$, $\lim_{x\to c} f(x) = L$ if and only if*

$$\lim_{x\to c^+} f(x) = L \text{ and } \lim_{x\to c^-} f(x) = L.$$

## Classification of discontinuities

*Given $f : A \to \mathbb{R}$ and a point $c \in A$, if $f$ is not continuous at $c$, then $c$ falls into one of the following categories:*

i. *If $\lim_{x\to c} f(x)$ exists, but has a value different from $f(c)$, then $f$ has a* ***removable discontinuity*** *at $c$.*
ii. *If $\lim_{x\to c} f(x)$ does not exist because $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, then $f$ has a* ***jump discontinuity*** *at $c$.*
iii. *If $\lim_{x\to c} f(x)$ does not exist for some other reason, then $f$ has an* ***essential discontinuity*** *at $c$.*

# Characterizing sets of discontinuity

**Theorem ($D_f$ for monotone functions)**

*The only type of discontinuity that a monotone function can have is a jump discontinuity. Furthermore, $D_f$ is either finite or countable.*

**Definition**

Let $f$ be defined on $\mathbb{R}$, and let $\alpha > 0$. The function $f$ is **$\alpha$-continuous at $x \in \mathbb{R}$** if there exists a $\delta > 0$ such that for all $y, z \in (x - \delta, x + \delta)$ it follows that $|f(y) - f(z)| < \alpha$. We also define $D_f^\alpha$ to be the set of points where $f$ fails to be $\alpha$-continuous.

**Theorem ($D_f$ for an arbitrary function)**

*Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then, $D_f$ is an $F_\sigma$ set.*

- Since $\mathbb{I}$ is not $F_\sigma$, there is no function $f : \mathbb{R} \to \mathbb{R}$ that has $D_f = \mathbb{I}$.
- How can we construct a function $f$ such that $D_f$ is a given finite, countable or $F_\sigma$ set?