Lecture #14

MA 511, Introduction to Analysis

June 15, 2021

L'Hopital's Rule

Theorem

Let f, g be continuous on an interval containing a, and differentiable at all other points. If $f(a), g(a) = 0$ or $\lim_{x \to a} g(x) = \infty$, and $g'(x) \neq 0$ for $x \neq a$, then $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L$

- The other versions of L'Hopital's Rule we see in calculus can all be handled by algebraic limit theorems and the cases we have proven

Non-Differentiability

- We saw at the beginning of the chapter that differentiable implies continuous. We also saw that continuous does not imply differentiable
- Consider the following questions:
  1. Does continuous everywhere imply differentiable somewhere?
  2. Is an arbitrary combination of differentiable functions have to be differentiable?
  3. How special is the property of differentiability? How weird can (continuous) functions be?

Nerdy Math Meme

The Weierstrass Function

Definition

For the remained of this section, let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = |x|$ for $-1 \leq x \leq 1$ and the property that $h(x+2) = h(x)$ for all $x \in \mathbb{R}$

- Note: h is continuous and differentiable everywhere except for $x \in \mathbb{Z}$. Also, $0 \leq h \leq 1$

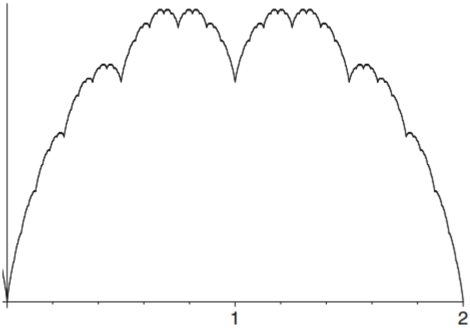
Definition

For all $n \in \mathbb{N}$, define $h_n(x) = \frac{1}{2^n} h(2^n x)$

Definition

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} h_n(x)$

The Weierstrass Function (Continued)



- An approximation of g on the interval $[0, 2]$
- Because $h(x+2) = h(x)$, $g(x+2) = g(x)$. The behavior of g everywhere is described by the behavior on $[0, 2]$

The Weierstrass Function (Continued)

Theorem

For all $n \in \mathbb{N}$, h_n is continuous everywhere, is differentiable everywhere except for $x = \frac{\mathbb{Z}}{2^n}$, and satisfies $h_n'(x) = h'(2^n x)$ where it is differentiable

Theorem

g is continuous everywhere on $\mathbb{R}$ but g is not differentiable anywhere on $\mathbb{R}$

Theorem

Let $0 < a < 1$ and b be such that $ab \geq 1$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$. f is continuous everywhere on $\mathbb{R}$ but is not differentiable anywhere on $\mathbb{R}$

- What properties of sequences/series of functions are preserved in the limit?
- Can we represent complicated functions as limits of simpler functions are use that as a tool for analysis?

Function Sequences

- Just as we can define sequences of Reals, we can define sequences of Real valued functions
- Ideally, we would like to recover as many properties of sequences as possible for the case of sequences of functions
- In this chapter, we will develop as many of these results as we can and use them to explore the behavior of more exotic functions

Definition (Pointwise Convergence of Functions)

For each $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function. We say that the sequence (f_n) converges pointwise on A to f if, for all $x \in A$, the sequence $(f_n(x))$ converges to $f(x)$

Continuity of the Limit Function

- It is NOT true that sequences of continuous functions converge to continuous functions
- If we attempt to prove such a "theorem", we may find the type of poor behavior which causes it to fail

Definition (Uniform Convergence of Functions)

For each $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function. We say that the sequence (f_n) converges uniformly on A to f if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in A$

Definition (Alternate Form of Pointwise Convergence)

For each $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function. We say that the sequence (f_n) converges pointwise on A to f if for all $\varepsilon > 0$ and for all $x \in A$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$

- We could even consider N in the pointwise case as a function $N : A \to \mathbb{N}$

Other Types of Convergence

Definition (Uniform Convergence on Compact Subsets)

For each $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function. We say that the sequence (f_n) converges uniformly on compact subsets of A to f if for all $K \subseteq A$ that are compact and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in K$

- This notion is more useful in complex analysis and functional analysis than real analysis
- We could define similar notions for pointwise convergence or for any other set property

Theorem

For each $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function. If (f_n) converges to f uniformly on A, then it converges to f uniformly on compact subsets of A. If (f_n) converges to f uniformly on compact subsets of A, then it converges to f pointwise on A