Lecture #16

MA 511, Introduction to Analysis

June 17, 2021

Properties of Power Series

Theorem

If $\sum_{n=1}^{\infty} a_n x^n$ converges pointwise on A, it converges uniformly on compact subsets of A

Theorem

If $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$, then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges on $(-R, R)$ as well. The convergence is uniform on compact subsets of $(-R, R)$

Theorem

Assume that $f(x) = \sum_{n=1}^{\infty} a_n x^n$ converges on an interval $A \subseteq \mathbb{R}$. f is continuous on A and infinitely differentiable on all intervals $(-R, R) \subseteq A$. The derivative of f is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

More Power Series Results

Theorem (Problem 6.5.7)

Let $\sum_{n=1}^{\infty} a_n x^n$ *be a power series such that* $a_n \neq 0$ *and* $\limsup \left|\frac{a_{n+1}}{a_n}\right| = L$*. The series converges on* $(-\frac{1}{L}, \frac{1}{L})$

Theorem (Problem 6.5.8)

$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ *if and only if* $a_n = b_n$ *for all* $n \in \mathbb{N}$

Theorem (Taylor's Formula for Coefficients)

If f *is an infinitely differentiable function defined on* $(-R, R)$ *and* f *can be represented as a power series,* $\sum_{n=1}^{\infty} a_n x^n$*, then*

$$a_n = \frac{f^{(n)}(0)}{n!}$$

- We still need to show that f can be represented this way

Errors of Taylor Polynomials

Definition (Taylor Polynomial)

Let f be N-times differentiable. For $0 \leq n \leq N$, we can define, the nth Taylor Polynomial of f as

$$S_n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Theorem (Lagrange Remainder Theorem)

Let f be $N+1$-times differentiable on $(-R, R)$ and $E_n(x) = f(x) - S_n(x)$. For all $x \neq 0$ in $(-R, R)$, there exists c such that $|c| < |x|$ and

$$E_N(x) = \frac{f^{N+1}(c)}{(N+1)!} x^{N+1}$$

- Note: Our choice of c does depend on x but for many (not all) functions we can bound $f^{(N+1)}(c)$

Moving the Center

Definition (Taylor Polynomial at a)

Let f be N-times differentiable and a in the domain of f. For $0 \leq n \leq N$, we can define, the nth Taylor Polynomial of f centered at a as

$$S_n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- We can define a similar shift for power series
- All of our established rules apply simply by shifting the center of all of our statements to a instead of 0

Cauchy Remainder Theorem

Theorem (Cauchy Remainder Theorem (from 6.6.9))

Let f *be* $N+1$*-times differentiable on* $(-R, R)$, $a \in (-R, R)$, *and define the following:*

$$S_N(x, a) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$E_N(x, a) = f(x) - S_N(x, a)$$

For $x \in (-R, R)$ *and* $x \neq 0$, *we can find* $c \in (0, x)$ *such that*

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

The Limitations of Taylor Series

- We have not actually shown that infinitely differentiable functions can be represented as power series because they can't

Theorem

The function f defined below is infinitely differentiable on $\mathbb{R}$, but the Taylor Series generated from f converges uniformly on all of $\mathbb{R}$ to the 0 function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{cases}$$

- It is easy to imagine a way to construct a function whose Taylor series converges to the Taylor series of any given function

A Different Perspective

Theorem

Let $C^\infty(A)$ denote the space of all infinitely differentiable functions $f : A \to \mathbb{R}$ for any $A \subseteq \mathbb{R}$. Let $\mathbb{R}^\mathbb{N}$ denote the space of all possible sequences of Real numbers. The map $T_a : C^\infty(\mathbb{R}) \to \mathbb{R}^\mathbb{N}$ defined for any $a \in A$ by

$$T_a(f) = \left(\frac{f(a)}{a!}, \frac{f'(a)}{1!}, \frac{f''(a)}{2!}, \ldots, \frac{f^{(n)}(a)}{n!}, \ldots \right)$$

is not injective

Analytic Functions

Definition (Analytic Functions)

$f \in C^{\infty}(A)$ is real analytic on A if, for all $x_0 \in A$, the Taylor series of f centered at x_0 converges to f on some interval $(x_0 - \delta, x_0 + \delta)$

Theorem

If we restrict T_a from the previous slide to the set of real analytic functions on $\mathbb{R}$, denoted $C^{\omega}(\mathbb{R})$, the map is injective

- We can make similar statements about any open intervals of $\mathbb{R}$. It may also hold for all open sets
- We can also restrict our codomain to make this make a bijection between vector spaces. We can either just use the range or figure out what sequence properties we need for the Taylor series to converge