Lecture #17

MA 511, Introduction to Analysis

June 21, 2021

Using Taylor Series for Differential Equations

- We know that not every C^∞ function can be represented by its Taylor series, but if it can, then we can differentiate and integrate term by term
- One strategy for solving differential equations is to:
  1. Show there is a unique solution.
  2. Assume that solution is analytic.
  3. Solve for the Taylor coefficients.
  4. Show the Taylor series converges to the only possible solution.
- There are many theorems to show analytic solutions exist just from the structure of the equation
- This is not necessarily the best method but it is a method

The Weierstrass Approximation Theorem

Theorem (Weierstrass Approximation Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be continuous. Given* $\varepsilon > 0$, *there exists a polynomial* $p(x)$ *such that*

$$|f(x) - p(x)| < \varepsilon$$

for all $x \in [a, b]$

- It is easy to see that this let us we can construct a sequence of polynomials that converge uniformly to f
- For analytic functions, we can just use Taylor polynomials. But what do we use for everything else?

Building Intuition

- First, let's try to approximate functions by piece-wise linear functions

Definition

A continuous function $\phi : [a, b] \to \mathbb{R}$ is polygonal if there exists a partition $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ such that ϕ is linear on each $[x_i, x_{i+1}]$

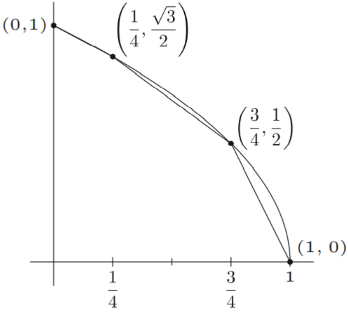

Figure 6.6: Polygonal approximation of $f(x) = \sqrt{1-x}$.

Polygonal Approximation Theorem

Theorem (Polygonal Approximation Theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$, there exists a polygonal function $\phi(x)$ such that

$$|f(x) - \phi(x)| < \varepsilon$$

for all $x \in [a, b]$

- Is this the functions constructed in our proof the approximation that requires the coarsest partition?
- We can easily define a polynomial to pass through the same points as the ones in our polygonal approximation. Does this approximate f?

Polynomial Interpolation

Definition (Interpolating Polynomial)

Let f be any function defines on an interval and x_k define a partition of the domain into M subintervals. The unique polynomial of degree $N-1$ agreeing with f on all x_k is

$$p(x) = \sum_{k=0}^{M} \left(\prod_{j=0\ ,\ j \neq k}^{M} \frac{x - x_j}{x_k - x_j} \right) f(x_k)$$

- While this is the *simplest* polynomial matching f at a given set of points it may not be the *best* approximation on that interval
- This method fails to approximate curves well when the points in the partition become close. As we add more points equally spaced, the values grows without bound in between them

The Way Forward

- We can approximate with polygonal functions so if we can figure out how to approximate those, the triangle inequality will do the rest
- The only complicated part seems to be the corners so if we can learn the trick for $|x|$, we can hopefully prove the result
- To approximate $|x|$, we actually need to look at $\sqrt{1-x}$ first.

Theorem (Exercises 6.7.4 - 6.7.6)

$\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n$ for $x \in [-1,1]$ and a_n defined by $a_0 = 1$ and

$$a_n = \prod_{k=1}^{n} \frac{2k-3}{2k}$$

Approximating $|x|$

Theorem

For any closed interval $[a, b]$ *and* $\varepsilon > 0$, *there is a polynomial* q *such that for all* $x \in [a, b]$

$$||x| - q(x)| < \varepsilon$$

Definition

Let $a \in [-1, 1]$ be fixed and define $h_a(x) = \frac{1}{2}\left(|x - a| + (x - a)\right)$

Theorem

Let ϕ *be a polygonal function on* $[a, b]$ *with partition points* a_k *for* $0 \leq k \leq n$. *There exist* b_k *such that*

$$\phi(x) = \phi(-1) + \sum_{k=0}^{n-1} b_k h_{a_k}(x)$$

Other Proofs

Definition

A Bernstein basis polynomial is a polynomial of the form

$$b_{v,n}(x) = \binom{n}{v} x^v (1-x)^{n-v}$$

A Bernstein polynomial is any polynomial which can be written in the form

$$B_n(x) = \sum_{v=0}^{n} \beta_v b_{v,n}(x)$$

Theorem (Bernstein Polynomial Approximation Theorem)

Let f be continuous on $[0,1]$. Define $P_n(x)$ by

$$P_n(x) = \sum_{v=0}^{n} f\left(\frac{v}{n}\right) b_{v,n}(x)$$

The sequence P_n converges to f uniformly.

Stone-Weierstrass Theorem

- The same approximation result holds for any compact set and any appropriate choice of continuous functions

Theorem (Stone-Weierstrass Theorem)

Let $K \subset \mathbb{R}$ be compact and $\mathcal{C}$ be a family of continuous functions such that

1. *$\mathcal{C}$ contains $f(x) = 1$*
2. *If $p, q \in \mathcal{C}$ and $c \in \mathbb{R}$, then $p + q, pq, cq \in \mathcal{C}$*
3. *If $x \neq y$, then there is $p \in \mathcal{C}$ such that $p(x) \neq p(y)$*

Any continuous function on K can be uniformly approximated by functions in $\mathcal{C}$