Lecture #18

MA 511, Introduction to Analysis

June 22, 2021

Riemann Sums

Definition (Upper and Lower Riemann Sums)

Let P be a partition of $[a, b]$, $m_k = \inf\{f(x) : x \in [x_k, x_{k+1}]\}$ and $M_k = \sup\{f(x) : x \in [x_k, x_{k+1}]\}$ for $0 \leq k \leq n-1$. The upper and lower Riemann sums of f with respect to P are defined as follows:

$$U(f, P) = \sum_{k=0}^{n-1} M_k(x_{k+1} - x_k)$$

$$L(f, P) = \sum_{k=0}^{n-1} m_k(x_{k+1} - x_k)$$

Definition (Refinement of a Partition)

A partition Q is a refinement of a partition P if $P \subseteq Q$

Changing Partitions

- For the following, we assume f is bounded so that the sums exist

Lemma

$L(f, P) \leq U(f, P)$ for all f and P

Lemma

If Q is a refinement of P, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq L(f, Q)$

Lemma

If P_1 and P_2 are partitions, then $L(f, P_1) \leq U(f, P_2)$

- The behavior is intuitively what we expect
- These results give us the behavior we want to eventually define integration as some sort of limiting process

Integration

Definition

Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. We define the upper and lower Riemann integrals of f over $[a, b]$ to be

$$U(f) = \inf \{U(f, P) : P \in \mathcal{P}\}$$
$$L(f) = \sup \{L(f, P) : P \in \mathcal{P}\}$$

Lemma

If f is bounded, then $L(f) \leq U(f)$

- It is difficult to imagine or work with $\mathcal{P}$, even though it is the most natural way to define U and L
- If f is not bounded, then we have terms that must be infinite in the definitions of $L(f, P)$, $U(f, P)$, or both. So, our definitions won't work

Riemann Integration

Definition (Riemann Integrability)

f is Riemann integrable on $[a, b]$ if $U(f) = L(f)$. We define $\int_a^b f(x)dx$ to be this value

- This definition makes intuitive sense, but we still have the issue of computing it
- What properties of f are necessary for integrability? What are sufficient?

Theorem (Integrability Criterion)

A bounded function f is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P_ε such that

$$U(f, P_\varepsilon) - L(U, P_\varepsilon) < \varepsilon$$

Integrability and Continuity

- $U(f, P) - L(f, P) = \sum_{k=0}^{n-1}(M_k - m_k)\Delta x_k$ so the integrability of f is related to its change over given intervals
- This sounds like uniform continuity

Theorem

If f is continuous on $[a, b]$, then it is integrable

Theorem

If f is bounded on $[a, b]$ and integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$. A similar result holds for subintervals $[a, c]$