Lecture #19

MA 511, Introduction to Analysis

June 23, 2021

Sequences of Partitions

Theorem (Sequential Criterion for Integrability)

If f is a bounded function, then f is integrable if and only if there is a sequence of partitions P_n such that

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$$

In this case

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$$

- We saw the idea of this proof in yesterday's examples and you are asked to provide the details for homework

Content Zero Sets and Integration

Definition (Content Zero Sets)

A set $A \subset [a, b]$ is said to have content zero if for all $\varepsilon > 0$, there is a family of open intervals $\{(c_1, d_1), ..., (c_n, d_n)\}$ which cover A and satisfy $\sum_{k=1}^{n} d_k - c_k \leq \varepsilon$

Theorem (Exercise 7.3.9)

If f is bounded and the set of discontinuities of f is content zero, then f is integrable.

- The idea behind this proof is similar to the approach for integrating the function $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ that we ended yesterday with

Properties of Integration

Theorem

Assume $f : [a, b] \to \mathbb{R}$ *is bounded and let* $c \in (a, b)$*.* f *is integrable on* $[a, b]$ *if and only if* f *is integrable on* $[a, c]$ *and* $[c, b]$*. If this is the case, then*

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem

Assume that f *and* g *are integrable on* $[a, b]$

1. *The function* $f + g$ *is integrable and* $\int_a^b (f + g) = \int_a^b f + \int_a^b g$
2. *For* $k \in \mathbb{R}$*, the function* kf *is integrable and* $\int_a^b (kf) = k \int_a^b f$
3. *If* $m \leq f \leq M$*, then* $m(b - a) \leq \int_a^b f \leq M(b - a)$
4. *If* $f(x) \leq g(x)$*, then* $\int_a^b f \leq \int_a^b g$
5. *The function* $|f|$ *is integrable and* $|\int_a^b f| \leq \int_a^b |f|$

Some Convenient Notation

Definition

Let f be integrable on $[a, b]$ and $c \in [a, b]$. We define the following notation

$$\int_b^a f = -\int_a^b f$$

$$\int_c^c f = 0$$

- This definition makes our theorem about breaking $[a, b]$ into $[a, c]$ and $[c, b]$ makes sense for any a, b, and c in a domain where f is integrable

Integration and Limits

Theorem (Integrable Limit Theorem)

Assume that $f_n \to f$ uniformly on $[a, b]$ and each f_n is integrable. Then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

- If $f_n \to f$ pointwise, then the result does not hold. We can make the limit not exist or make the equation false

The Fundamental Theorem of Calculus

Theorem (The Fundamental Theorem of Calculus)

1. *If* $f : [a,b] \to \mathbb{R}$ *is integrable and* $F : [a,b] \to \mathbb{R}$ *is differentiable with* $F'(x) = f(x)$ *for all* $x \in [a,b]$*, then*

$$\int_a^b f = F(b) - F(a)$$

2. *Let* $g : [a,b] \to \mathbb{R}$ *be integrable, and for* $x \in [a,b]$*, define the function* $G : [a,b] \to \mathbb{R}$ *by*

$$G(x) = \int_a^x g$$

   G *is continuous on* $[a,b]$*. If* g *is continuous at* $c \in [a,b]$*, then* G *is differentiable at* c *and* $G'(c) = g(c)$