Lecture #2

MA 511, Introduction to Analysis

May 25, 2021

The Axiom of Completeness

- $\mathbb{Q}$ is an **ordered field**. The natural order $<$ is such that for rationals r and s exactly one of the following to be true: $r < s$, $r = s$, or $r > s$.

Definition

A **field** is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the distributive property $a(b + c) = ab + ac$. There must be an additive identity and a multiplicative identity. All elements must have an additive inverse and all nonzero elements must have a multiplicative inverse.

- $\mathbb{R}$ should be an ordered field, which contains and extends $\mathbb{Q}$, but what exactly is a real number and how can we "plug the gaps" in $\mathbb{Q}$?

Axiom of Completeness

Every nonempty set of real numbers that is bounded above has a least upper bound.

Least Upper Bounds and Greatest Lower Bounds

Definition

A real number $s = \sup A$ is the **least upper bound** (or **supremum**) for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

i. s is an upper bound for A
ii. if b is any upper bound for A then $s \leq b$

If $s \in A$ it is called the **maximum** of A.

Definition

A real number $i = \inf A$ is the **greatest lower bound** (or **infimum**) for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

i. i is an lower bound for A
ii. if b is any lower bound for A then $i \geq b$

If $i \in A$ it is called the **minimum** of A.

- If they exist, are $\sup A$ and $\inf A$ unique?

Consequences of Completeness

- The first result that we can prove perhaps better expresses that $\mathbb{R}$ contains no "gaps."

Theorem (Nested Interval Property)

For each $n \in \mathbb{N}$, assume we are given a closed interval:

$$I_n = [a_b, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$$

Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection, i.e. $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

- We will see later that the Nested Interval Property could have been our fundamental axiom of the real numbers (provided that we also assumed the Archimedean Property).

Density of $\mathbb{Q}$ in $\mathbb{R}$

- $\mathbb{R}$ is an extension of $\mathbb{Q}$, which is an extension of $\mathbb{N}$, but how do $\mathbb{N}$ and $\mathbb{Q}$ sit inside $\mathbb{R}$?

Theorem (Archimedean Property)

i. *Given any number $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ satisfying $n > x$.*
ii. *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.*

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Corollary

Given any two real numbers a and b, there exists an irrational number t satisfying $a < t < b$.

The Existence of Square Roots

Theorem

There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

- Similarly, we can show $\sqrt{x}$ exists for any $x \geq 0$.
- Using the binomial theorem to expand:

$$\left(\alpha + \frac{1}{n}\right)^m = \sum_{k=0}^{m} \binom{m}{k} \frac{\alpha^{m-k}}{n^k} = \alpha^m + m\frac{\alpha^{m-1}}{n} + \cdots + \frac{1}{n^m}$$

  we can also show that $\sqrt[m]{x}$ exists for arbitrary values of $m \in \mathbb{N}$.
- Are the rationals $\mathbb{Q}$ and the irrationals $\mathbb{I}$ each **closed under addition and multiplication**?
- If $r \in \mathbb{Q}$ and $t \in \mathbb{I}$, what can we say about $a + t$ and at (assuming $a \neq 0$)?
- What are the "proportions" of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$?

Cardinality

- What is the "size" of $\mathbb{Q}$ anyway?

Definition

A function $f : A \to B$ is **one-to-one** (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is **onto** if, given any $b \in B$, it is possible to find an element of $a \in A$ for which $f(a) = b$. A function that is both one-to-one and onto is called a **one-to-one correspondence**.

Definition

The **cardinality** of a set refers is a measure of its size. The set A has the same cardinality as B if there exists a one-to-one correspondence $f : A \to B$. In this case, we write $A \sim B$.

Example: If E is the set of even natural numbers, then $E \sim \mathbb{N} \sim \mathbb{Z}$. If (a, b) is any interval of real numbers, then $(a, b) \sim \mathbb{R}$.