Lecture #22

MA 511, Introduction to Analysis

June 29, 2021

Trigonometric Series

Definition (Trigonometric/Fourier Series)

A series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

is called a trigonometric series

- If we consider complex numbers and use Euler's equation, $e^{ix} = \cos(x) + i\sin(x)$, we have an even nicer formula

$$\sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$$

- We can recover a_n and b_n via the formula $a_n = \frac{\alpha_n + \alpha_{-n}}{2}$ and $b_n = \frac{\alpha_n - \alpha_{-n}}{2}$

Periodic Functions

Definition (Periodic Functions)

A function $f : \mathbb{R} \to \mathbb{R}$ is periodic if there exists $\xi \in \mathbb{R}^+$ such that for all $x \in \mathbb{R}$

$$f(x) = f(x + \xi)$$

- A simple calculation shows that if ξ satisfies the equation above, then so does $n\xi$ for all $n \in \mathbb{N}$

Definition (Period of a Function)

If f is periodic, the period of f is the smallest value ξ such that $f(x) = f(x + \xi)$ if such a value exists

- We can just as easily think of these as functions on $(-\frac{\xi}{2}, \frac{\xi}{2}]$ extended periodically

Fourier's Claim

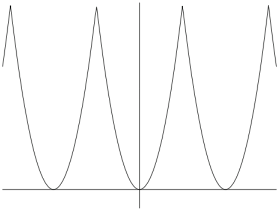

Figure 8.2: $f(x) = x^2$ **over** $(-\pi, \pi]$**, extended to be** 2π**-periodic.**

- The result Fourier claimed is "Thus, there is no function... which cannot be expressed by a trigonometric series"
- In more modern terminology, "All periodic functions can be written as limits of trigonometric series"
- The conditions needed for convergence in most cases are much milder than the conditions needed for Taylor series
- What conditions do we need to get convergence? Is there a formula for a_n and b_n?

Inner Product Spaces

- We know that on $\mathbb{R}^n$, the dot product takes 2 vectors and returns a real number
- We also know that this can be used to find angles between vectors and the projections from 1 vector onto another
- The idea of an inner product is to generalize this

Definition (Inner Product)

An inner product on a real vector space, V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ which satisfies the following properties for all $x, y, z \in V$ and $a \in \mathbb{R}$

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle ax, y \rangle = a\langle x, y \rangle$
3. $\langle x, y \rangle = \langle y, x \rangle$
4. $\langle x, x \rangle > 0$ if $x \neq 0$

Orthonormal Bases in Inner Product Spaces

Definition (Orthogonality)

Vectors f and g in an inner product space are said to be orthogonal if $\langle f, g \rangle = 0$

Definition (Normality)

A vector f in an inner product space is said to be normal if $\langle f, f \rangle = 1$

Definition (Basis)

A collection of vectors, $\mathcal{B}$, in an inner product space, H, is a basis if all $h \in H$ can be written as a (countable) linear combination of vectors in $\mathcal{B}$ and $\mathcal{B}$ is linearly independent

- These are the normal properties from $\mathbb{R}^n$ but rewritten for ANY inner product space

L^2 Space

- Technically, we need Lesbegue measure to define L^2 space

Definition (L^2 Space)

$L^2([a,b])$ is the space of functions* for which $\int_a^b f^2(x)dx < \infty$. The map

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

is an inner product on L^2

- Using this inner product, we can define the "length" of functions, the "distance" between them, and the "angle" between them in similar ways as our usual vectors using the dot product
- Functional analysis is the relevant field for the analysis of spaces like these

Rephrasing Fourier's Claim

- Using this new terminology, we can rewrite Fourier's claim in a functional analysis framing

Theorem

The set of functions $\{1, \cos(nx), \sin(nx)\}_{n=1}^{\infty}$ *forms a basis for the space* $L^2((-\pi, \pi])$

Corollary

The set of functions $\left\{\frac{1}{2\pi}, \frac{1}{\pi}\cos(nx), \frac{1}{\pi}\sin(nx)\right\}_{n=1}^{\infty}$ *forms an orthonormal basis for the space* $L^2((-\pi, \pi])$

- Technically, Fourier's claim only really states that the trigonometric series converges pointwise. We will also show the convergence works in other ways

Fourier Coefficients

- If we believe our theorem, then we can use the vector projection formula to get our coefficients.

$$f = \sum_{b \in \mathcal{B}} \frac{\langle f, b \rangle}{\langle b, b \rangle} b$$

- Alternatively, standard algebra and integration can also give our formula

Theorem

The Fourier coefficients for f are as follows:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Example 1

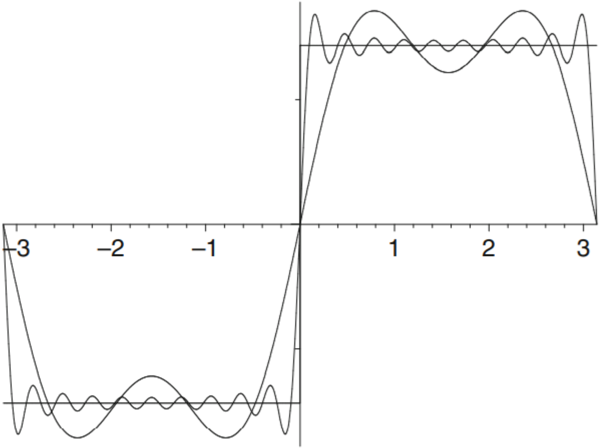

Figure 8.3: f, S_4, **and** S_{20} **on** $[-\pi, \pi]$.

Riemann-Lebesgue Lemma

- We will be treating functions as being periodically extended to all of $\mathbb{R}$ unless otherwise stated
- Properties like continuity are taken to mean that their periodic extensions have these properties

Theorem (Riemann-Lebesgue Lemma)

Let h be a continuous function on $(-\pi, \pi]$. Then as $n \to \infty$, we get

$$\int_{-\pi}^{\pi} h(x)\cos(nx)dx \to 0$$

$$\int_{-\pi}^{\pi} h(x)\sin(nx)dx \to 0$$

Conditions for Convergence

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$f(x)$	$=$
bounded	pointwise convergence
integrable	uniform convergence
continuous	L^2 convergence
differentiable	Cesaro mean convergence
f' continuous	

- As we have mentioned, we need different conditions on f to make the corresponding Fourier series converge in different ways
- L^2 convergence actually only needs a function to be **Lebesgue** integrable

Additional Convergence Theorems

Theorem

Let f be continuous on $(-\pi, \pi]$ and differentiable at c. Then, $\lim_{n\to\infty} S_n(c) = f(c)$

Theorem

Let f be continuous on $(-\pi, \pi]$. Then $\sigma_n(x) \to f(x)$ uniformly where

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(x)$$