Lecture #23

MA 511, Introduction to Analysis

June 30, 2021

A Review of the Reals

- When we first defined the Real numbers we discussed a few important properties
  1. $\mathbb{R}$ is an ordered field
  2. $\mathbb{R}$ contains $\mathbb{Q}$
  3. $\mathbb{R}$ is complete (Axiom of Completeness)
- $\mathbb{R}$ has some other interesting properties worth exploring in generality
  4. $\mathbb{R}$ is a topological space
  5. $\mathbb{R}$ is a (real) normed vector space
  6. $\mathbb{R}$ is a (real) inner product space
  7. $\mathbb{R}$ is a measure space
- Each of these properties can be explored in more generality
- Many of these properties automatically give other properties

$$\text{I.P.S} \implies \text{N.V.S} \implies \text{M.S} \implies \text{T.S}$$

Our Focus for This Course

- Most of what we have done has been focused on $\mathbb{R}$ as a metric space
- We have discussed toplogical properties but these discussions always used the frame of metric spaces
- We have used the ordered field properties of $\mathbb{R}$ implicitly but only to help our work on $\mathbb{R}$ as a metric space
- Our discussions of functions and sequences thereof were all in terms of metric space properties as well

Metric Spaces

- We can generalize the properties of distances on $\mathbb{R}$ to any set with an appropriately defined notion of distance

Definition (Metric Space)

A metric space is a set, X, along with a map, $d : X \times X \to \mathbb{R}$, called the metric, which satisfies the following properties for all $x, y \in X$

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \implies x = y$
2. $d(x, y) = d(y, x)$
3. d satisfies the triangle inequality: For all $z \in X$, we get $d(x, y) \leq d(x, z) + d(z, y)$

- This matches all of our intuitions about what "distance" should mean

Sequences in Metric Spaces

Definition (Convergent Sequences)

A sequence, (x_n), of points in the metric space (X, d) is said to converge to a limit x if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$

Definition (Cauchy Sequences)

A sequence, (x_n), of points in the metric space (X, d) is said to be Cauchy if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$

Theorem

Any sequence which converges is a Cauchy sequence

Definition (Completeness)

A metric space is complete if all Cauchy sequences converge to a point in X

A Brief Introduction to Topology

Definition (Topology on a Set)

A topology on a set, X, is a collection of subsets, $\mathcal{T}$, such that for all $A_i \in \mathcal{T}$, the following hold

1. $\mathcal{T}$ contains both X and $\emptyset$
2. $\mathcal{T}$ contains $\bigcup_{i \in I} A_i$ for any index I
3. $\mathcal{T}$ contains $\bigcap_{j=1}^{n} A_j$

$O \subseteq X$ is called open if $O \in \mathcal{T}$. $C \subseteq X$ is called closed if C^c is open

Definition (Basis for a Topology)

A basis, $\mathcal{B}$, for a topology is a collection of sets $B_i \in \mathcal{T}$ such that all sets $A \in \mathcal{T}$ can be written as $A = \bigcup_{i \in I} B_i$ or $A = \bigcap_{j=1}^{n} B_j$

- Having open sets allows us to discuss properties like continuity in more general terms

The Metric Topology

Definition (ε-neighborhoods)

Let (X, d) be a metric space. We define the ε-neighborhood of x as $V_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$

Theorem (The Metric Topology)

The set of all ε-neighborhoods of all points is a basis for a topology on X. We call this the metric topology

- Defining the metric topology like this unifies the topological definitions of properties with the metric space notions

Properties of the Metric Topology

Theorem

A set, A, is open if and only if for all $x \in A$, there is $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq A$

Definition (Limit Points)

x is a limit point of A if $V_\varepsilon(x) \cap A \neq \{x\}$

Theorem

x is a limit point of A if and only if there exists a sequence of points $(x_n) \in A \setminus \{x\}$ such that $(x_n) \to x$

Theorem

A set, A, is closed if and only if A contains all of its limits points

Compactness

Definition (Compact Sets)

A set A, is compact in the sequential/metric sense if every sequence (x_n) contained in A has a subsequence (x_{n_k}) which converges to a point in A

Definition (Boundedness)

A set, A, in a metric space is bounded if for all $x, y \in A$, there exists $M > 0$ such that $d(x, y) \leq M$

Theorem

Any compact set in a metric space must be closed and bounded

- Notice that this theorem is NOT an if and only if statement
- Just like many functions had nice properties on compact subsets of $\mathbb{R}$, functions on compact sets of more general spaces often have nice properties

Maps Between Spaces

Definition (Continuity)

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is continuous at $x \in X$ in the metric sense if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x') < \delta$ implies that $d_Y(f(x), f(x')) < \varepsilon$

- We can define uniform continuity as well as notions of convergence of sequences of maps in a similar way as with $\mathbb{R}$
- We cannot define series of points or of functions because we do not have a way of adding points together
- Many theorems about the behavior of functions from $\mathbb{R}$ carry over directly to (complete) metric spaces as well
- Anything which required the statement "closed and bounded implies compact" is likely to fail (in ∞-dimensional spaces in particular)

Density of Sets

Definition (Closure of a Set)

The closure of A is defined to be

$$\overline{A} = \{x \in X : a \text{ is a limit points of } A\}$$

Definition (Dense Sets)

A set, A, is dense in (X, d) if $\overline{A} = X$.

Definition (Interior of a Set)

The interior of a set, A, is the set

$$A^\circ = \{x \in A : \exists \varepsilon > 0 \text{ such that } V_\varepsilon(x) \subseteq A\}$$

Definition (Nowhere-Dense Sets)

A set, A, is nowhere-dense if $\overline{A}^\circ = \emptyset$

The Baire Category Theorem

Theorem

Let $\{O_n\}_{n=1}^{\infty}$ *be a countable collection of open, dense sets. Then,*

$$\bigcap_{n=1}^{\infty} O_n \neq \emptyset$$

Theorem

A set, A, is nowhere dense if and only if $\overline{A}^c$ *is dense*

Theorem (Baire Category Theorem)

If (X, d) is a complete metric space, then X cannot be written as a countable union of nowhere-dense sets

- So, in all metric spaces, there is a natural categorization of sets by whether they can or cannot be expressed as the countable union of nowhere-dense sets

Differentiability is Rare

Theorem

The set of functions,

$$D = \{f \in C[0,1] : f \text{ is differentiable at at least 1 point}\}$$

is of first category in $(C[0,1], d_\infty)$

- One key idea used in the proof is the equivalence between the statements "$|f(x) - p(x)| < \varepsilon$ for all $x \in [0,1]$" and "$d_\infty(f,p) < \varepsilon$" for p any continuous function
- This gives us a new perspective on the Stone-Weierstrass theorem (and the other approximation theorems). Any set of continuous functions satisfying the SW theorem hypotheses must be dense in $(C[0,1], d_\infty)$

Topological Notions

Definition (Compactness)

A set, A, is compact in the topological sense if every cover of A by open sets has a finite subcover

Definition (Continuity)

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is continuous at $x \in X$ in the topological sense if for all $B \in \mathcal{T}_Y$ such that $f(x) \in B$, there is a set $A \in \mathcal{T}_X$ such that $x \in A$ and $f(A) \subseteq B$

Definition (Density)

A set, A, is dense in $(X, \mathcal{T})$ in the topological sense if for all $B \in \mathcal{T} \setminus \{\emptyset\}$, $A \cap B \neq \emptyset$

Topological Notions (Continued)

Definition (Interior)

The interior of a set, A, is the set

$$A^{\circ} = \{x \in A : \exists B \in \mathcal{T} \text{ such that } x \in B \text{ and } b \subseteq A\}$$

Definition (Nowhere-Density)

A set, A, is nowhere-dense if $\overline{A}^{\circ} = \emptyset$

Theorem

For metric spaces with the respective metric topologies, the topological and metric definitions of the previous properties are equivalent.