Lecture #24

MA 511, Introduction to Analysis

July 1, 2021

Euler's Sum

- We have seen but never proven that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

- The prof in the book relies on expressing $\sin(x)$ as an infinite sum AND and infinite product

$$\begin{aligned}\sin(x) &= -\sum_{n=1}^{\infty} \frac{(-x)^n}{(2n-1)!} \\ &= x \prod_{n=1}^{\infty} \left(1 - \frac{x}{n\pi}\right)\left(1 + \frac{x}{n\pi}\right)\end{aligned}$$

- There are dozens of proofs using all sorts of analytic techniques. Some are purely based on Taylor series and convergence properties while other use multivariable calculus
- 3blue1brown has an interesting informal explanation based around a geometric construction and the inverse square law for light intensity

Wallis's Product

- Wallis's product comes from the spacial case of the $\sin(x)$ product formula at $\frac{\pi}{2}$

$$\prod_{n=1}^{\infty}\left(\frac{4n^2}{4n^2-1}\right)=\frac{\pi}{2}$$

- We define $b_n=\int_0^{\frac{\pi}{2}}\sin^n(x)dx$ and use integration by parts and induction to prove that

$$b_0=\frac{\pi}{2}$$
$$b_1=1$$
$$b_n=\frac{n-1}{n}b_{n-2}$$

Wallis's Product (Continued)

- Some algebra/induction shows that

$$b_n = \begin{cases} b_0 \prod_{k=1}^{\frac{n}{2}} \left(\frac{k-1}{k}\right) & \text{if } n \text{ even} \\ b_1 \prod_{k=1}^{\frac{n-1}{2}} \left(\frac{k-1}{k}\right) & \text{if } n \text{ odd} \end{cases}$$

- If we look at $\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}}$, some algebra will reveal that the limit is 1
- Algebra will also reveal that $\frac{b_0}{b_1} \lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}}$ is the Wallis product which must therefore have value $\frac{\pi}{2}$

The $\arcsin(x)$ Formula

- We will also need the Taylor series for $\arcsin(x)$ for this proof.
- $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$ so if we can find the Taylor series for $\frac{1}{\sqrt{1-x^2}}$ we can apply term by term integration to find the series for $\arcsin(x)$
- We can inductively show that the coefficients are $a_0 = 1$ and $a_n = \frac{(2n)!}{2^{2n}(n!)^2}$
- Some algebra can relate these terms to the Wallis product again, giving the following limit

$$\lim_{n\to\infty} \frac{1}{a_n\sqrt{n}} = \sqrt{\pi}$$

- We need to show that this converges on $(-1, 1)$, but our usual versions of the remainder theorem do not work. We will need a different version

The arcsin(x) Formula (Continued)

Theorem (Integral Form of the Remainder Theorem)

Let f be $N+1$ times continuously differentiable on $(-R, R)$ and $E_N(x)$ be the usual Taylor series error function. Then

$$E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N dt$$

- The book has a description of how to prove this that you can look at if you are interested
- We claim that this form will let us show convergence of the formula for $\frac{1}{\sqrt{1-x^2}}$ on $(-1,1)$
- With that being the case, we can conclude that for $-1 < x < 1$

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} x^{2n+1}$$

Evaluating the Sum

- $\theta = \arcsin(\sin(\theta))$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$
- Using out series for $\arcsin(x)$ we get

$$\theta = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} \sin^{2n+1}(\theta)$$

- We can related this back to our work on the Wallis product by integrating over $[0, \frac{\pi}{2}]$

$$\int_0^{\frac{\pi}{2}} \theta d\theta = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} b_{2n+1}$$

Evaluating the Sum (Continued)

- After some cancellation and some integration, we arrive at the formula

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

- From this and some clever algebra, we get our result

$$\frac{1}{(2k)^2} = \frac{1}{4}\frac{1}{k^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$\frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}$$

The Riemann Zeta Function

- We know that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $s \geq 2$ by the comparison test and that it diverges for $s = 1$
- It seems natural to ask what $\sum_{n=1}^{\infty} \frac{1}{n^s}$ looks like as a function of s
- This is the beginning of the definition of the Riemann Zeta function
- Some complex analysis results show that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $\Re(s) > 1$ so we can define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ on this domain
- We can use complex analytic continuation (expressing this function as a power series) we expand the domain to be $\mathbb{C} \setminus \{1\}$
- $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ is prime}} \left(\frac{1}{1-p^{-s}} \right)$ so the behavior of this function somehow encodes information about the behavior of prime numbers
- The Riemann Hypothesis states that $\zeta(s) = 0$ only if $s = -2n$ or if $\Re s = \frac{1}{2}$. If you can prove this, you get a million dollars and your name in every analysis and number theory textbook written for the rest of time