Lecture #3

MA 511, Introduction to Analysis

May 26, 2021

Countable Sets

Definition

A set A is **finite** if $A \sim \{1, \ldots, n\}$ for some n. A set A is **countable** if $\mathbb{N} \sim A$. An infinite set that is not countable is called an **uncountable** set.

Theorem

i *The set $\mathbb{Q}$ is countable.*

ii *The set $\mathbb{R}$ is uncountable.*

Theorem

If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem

i *If $A_1, \ldots, A_m$ are each countable sets then the union $A_1 \cup \cdots \cup A_m$ is countable.*

ii *If A_n is a countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.*

Cantor's Theorem

Theorem

The open interval $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ *is uncountable.*

- There is a hierarchy of infinite sets that continues well beyond the continuum of $\mathbb{R}$.

Definition

Given a set A, the **power set** $P(A)$ (or 2^A) refers to the collection of all subsets of A. (Note that $P(A)$ is itself a set whose elements are sets.)

Theorem (Cantor's Theorem)

Given any set A*, there does not exist a function* $f : A \to P(A)$ *that is onto.*

Example: $P(\mathbb{N})$ is uncountable and in fact $P(\mathbb{N}) \sim \mathbb{R}$.

Equivalence Relations

- The relationship of having the same cardinality is an equivalence relation.

Definition

A binary relation $\sim$ on a set A is an **equivalence relation** if and only if for all a, b and c in A:

i. $a \sim a$ (reflexivity)
ii. $a \sim b$ if and only if $b \sim a$ (symmetry)
iii. if $a \sim b$ and $b \sim c$, then $a \sim c$ (transitivity)

Equivalence relations provide partitions of the set A into **equivalence classes** of the form $[a] = \{x \in A : x \sim a\}$.

Example: Equality $(=)$ on $\mathbb{R}$ is an equivalence relation. Having the same parity (even or odd) is an equivalence relation on $\mathbb{Z}$. Having the same remainder modulo n is an equivalence relation on $\mathbb{Z}$

Cardinal Numbers

- $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have the same cardinality and are hence in the same equivalence class. They all have the same "cardinal number" $\aleph_0$.

Definition

Roughly speaking the cardinal number of A, denoted $\operatorname{card} A$ is the equivalence class of all sets which have the same cardinality as A. That is, $\operatorname{card} A = \operatorname{card} B$ if and only if $A \sim B$. (Note that this definition poses problems with set theory and $\operatorname{card} A$ should actually be defined as a particular representative of $[A]$ that can always be uniquely determined.)

- We can order the cardinals, by setting $\operatorname{card} A \leq \operatorname{card} B$ whenever there is a one-to-one map from A to B. If it is also the case that $A \not\sim B$, then we write $\operatorname{card} A < \operatorname{card} B$.
- Cantor's Theorem $\Rightarrow \operatorname{card} A < \operatorname{card}(P(A)) < \operatorname{card}(P(P(A))) < \cdots$
- Does there exist a set A such that $\operatorname{card} \mathbb{N} < \operatorname{card} A < \operatorname{card} \mathbb{R}$?

Sequences and Convergence

- Our intuitions are severely broken when manipulating infinite series, so we need to develop a logically rigorous theory of sequences and series, if we hope to prove things about them.

Definition

A **sequence** is a function whose domain is $\mathbb{N}$ (or sometimes $\mathbb{N} \cup \{0\}$). Given $f : \mathbb{N} \to \mathbb{R}$, $f(n)$ is the nth number in an infinite ordered list.

Example: $(1, \frac{1}{2}, \frac{1}{3}, \dots)$, $(\frac{1+n}{n})_{n=1}^{\infty}$, and (a_n) where $a_1 = 1$ and $a_n = 3a_{n-1} + 1$ for $n > 1$ are all ways to describe a sequence.

Definition

A sequence (a_n) **converges** to a real number a if, for every positive number $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \varepsilon$. We write either $\lim a_n = a$, $\lim_{n\to\infty} a_n = a$ or $(a_n) \to a$.

- N depends on the choice of ε!