Lecture #4

MA 511, Introduction to Analysis

May 27, 2021

Topological version of convergence

Definition

Given a real number $a \in \mathbb{R}$ and a positive number $\varepsilon > 0$, the set:

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$$

is called the ε-neighborhood of a.

Definition (Topological version of convergence)

A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\varepsilon(a)$. In other words every ε-neighborhood contains all but a finite number of the terms of (a_n).

Theorem (Uniqueness of limits)

The limit of a sequence, when it exists must be unique.

Template for a proof that $(x_n) \to x$

- The quantifies "for all" ($\forall$) and "there exists" ($\exists$) make the definition of convergence complicated, so here is a basic outline for how every convergence proof should be presented.

Template for a proof that $(x_n) \to x$

1. *"Let $\varepsilon > 0$ be arbitrary."*
2. *Demonstrate a choice for $N \in \mathbb{N}$. This step usually requires the most work, almost all of which is done prior to writing the formal proof.*
3. *Now, show that N actually works.*
4. *"Assume that $n \geq N$."*
5. *With N well chosen, it should be possible to derive the inequality $|x_n - x| < \varepsilon$.*

Definition

A sequence that does not converge is said to **diverge**.

Algebraic limit theorem

Definition

A sequence (x_n) is **bounded** if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem

Every convergent sequence is bounded.

Theorem (Algebraic limit theorem)

Let $\lim a_n = a$ *and* $\lim b_n = b$. *Then, we have:*

- i $\lim(ca_n) = ca$, *for all* $c \in \mathbb{R}$
- ii $\lim(a_n + b_n) = a + b$
- iii $\lim(a_n b_n) = ab$
- iv $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, *provided* $b \neq 0$

Limit order theorem

Theorem (Limit order theorem)

Assume that $\lim a_n = a$ *and* $\lim b_n = b$.

- i *If* $a_n \geq 0$ *for all* $n \in N$, *then* $a \geq 0$.
- ii *If* $a_n \leq b_n$ *for all* $n \in N$, *then* $a \leq b$.
- iii *If there exists* $c \in \mathbb{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $c \leq b$. *Similarly if* $a_n \leq c$ *for all* $n \in \mathbb{N}$, *then* $a \leq c$.

- In analysis we typically only care about the "tails" of sequences.
- In part i above, we can replace $a_n \geq 0$ for all $n \in \mathbb{N}$ with $a_n \geq 0$ for all $n \geq N$ for some $N \in \mathbb{N}$. In this case, we might say that (a_n) is "eventually" nonnegative.

Theorem (Squeeze theorem)

If $x_n \leq y_n \leq z_n$ *for all* $n \in \mathbb{N}$ *and if* $\lim x_n = \lim z_n = l$, *then* $\lim y_n = l$ *as well.*

Monotone convergence theorem

- All convergent sequences are bounded, but do all bounded sequences converge?

Definition

A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotone** if it is either decreasing or increasing.

Theorem (Monotone convergence theorem)

If a sequence is monotone and bounded, then it converges.

- Note that MCT tells us that a sequence converges without mention of its actual limit!