Lecture #5

MA 511, Introduction to Analysis

June 1, 2021

Convergence of a series

Definition

Let (b_n) be a sequence. An **infinite series** is a formal expression of the form:

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + \cdots$$

We define the corresponding **sequence of partial sums** (s_m) to be:

$$s_m = \sum_{n=1}^{m} b_n = b_1 + b_2 + \cdots + b_m$$

We say that the series $\sum_{n=1}^{\infty} b_n$ **converges** to B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by MCT although we cannot yet find what it converges to.

Cauchy condensation test

Example: The "harmonic series" $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. The argument for why it diverges can be generalized to a large class of infinite series, as follows.

Theorem (Cauchy condensation test)

Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series

$$\sum_{n=1}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots$$

converges.

Corollary

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Subsequences and convergence

Definition

Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \dots)$$

is called a **subsequence** of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the sequence.

Theorem

Subsequences of a convergent sequence converge to the same limit as the original sequence.

Corollary

For a real number $b \in \mathbb{R}$, $(b^n) \to 0$ *if and only if* $-1 < b < 1$

Bolzano-Weierstrass theorem

- The contrapositive of the previous theorem can serve as a useful criterion for the divergence of a sequence.

Corollary (Divergence criterion)

If two subsequences converge to two different limits, then the original sequence diverges.

Example: For $(1, -1, 1, -1, 1, -1, \dots)$, the even indexed terms converge to -1 and the odd indexed terms to 1, so the sequence diverges.

- When can we find a convergent subsequence? The Bolzano-Weierstrass theorem offers a partial answer.

Theorem (Bolzano-Weierstrass theorem)

Every bounded sequence contains a convergent subsequence.

The Cauchy criterion

Definition

A sequence (a_n) is called a **Cauchy sequence** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \varepsilon$.

Theorem

Every convergent sequence is a Cauchy sequence.

Lemma

Cauchy sequences are bounded.

Theorem (Cauchy criterion)

A sequence converges if and only if it is a Cauchy sequence.

Completeness of $\mathbb{R}$ revisited

- So, far we have proved the following implications:

$$\text{AoC} \Rightarrow \begin{cases} \text{NIP} \Rightarrow \text{BW} \Rightarrow \text{CC} \\ \text{MCT} \end{cases}$$

- This is not the whole story! It turns out that:

$$\text{AoC} \Leftrightarrow \text{MCT} \Leftrightarrow \text{BW}$$

- Neither NIP nor CC are enough to prove the Archimedean property, but if we assume the Archimedean property, then we have:

$$\text{AoC} \Leftrightarrow \text{NIP} \Leftrightarrow \text{MCT} \Leftrightarrow \text{BW} \Leftrightarrow \text{CC}$$

- So any of these could serve as our defining axiom of the real numbers! Each asserts the completeness of $\mathbb{R}$ in its own language.
- Since we have an example of an ordered field that is not complete, i.e. $\mathbb{Q}$, it is impossible to prove any of these using only the field and order properties.