Lecture #6

MA 511, Introduction to Analysis

June 2, 2021

A construction of $\mathbb{R}$ from Cauchy sequences

- A set of axioms is meaningless if nothing satisfies it, so we will now show that such a field exists.
- We cannot prove that Cauchy sequences converge using only the field and order properties, so a standard procedure to force all Cauchy sequences to converge is adding new points to the metric space in a process called **completion**.

Definition

First of all consider the set:

$$C_{\mathbb{Q}} = \{(x_n) : (x_n) \text{ is a Cauchy sequence in } \mathbb{Q}\}$$

We can define an equivalence relation on $C_{\mathbb{Q}}$ as follows:

$$(x_n) \sim (y_n) \text{ if and only if } (x_n - y_n) \to 0$$

Now, define $\mathcal{R}$ to be the set of equivalence classes of $C_{\mathbb{Q}}$ with respect to this equivalence relation, which we also call the completion of $\mathbb{Q}$.

Operations on $\mathcal{R}$

Proposition

The relation $(x_n) \sim (y_n)$ if and only if $(x_n - y_n) \to 0$ on $C_{\mathbb{Q}}$ is an equivalence relation.

Proposition

Equivalence classes are either equal or disjoint. Thus, given an equivalence relation $\sim$ on a set S, the equivalence classes form a partition of S.

Definition

We can define addition and multiplication on $\mathcal{R}$ as follows:

$$[(x_n)] + [(y_n)] = [(x_n + y_n)] \text{ and } [(x_n)] \cdot [(y_n)] = [(x_n \cdot y_n)]$$

Now, $\alpha \in \mathcal{R}$ is **positive** if there is some eventually positive sequence (x_n) such that $\alpha = [(x_n)]$. Then, we can define an order on $\mathbb{R}$ as follows:

$$[(x_n)] < [(y_n)] \text{ if } [(y_n - x_n)] \text{ is positive}$$

Field properties

Proposition

The operations of addition and multiplication on $\mathcal{R}$ are well-defined.

Proposition (Field properties)

For all $a, b, c \in \mathcal{R}$, we have:

1. $a + (b + c) = (a + b) + c$ *and* $a(bc) = (ab)c$
2. $a + b = b + a$ *and* $ab = ba$
3. $a + 0 = a$ *and* $a \cdot 1 = a$
4. *There exists* $a' \in \mathcal{R}$ *such that* $a + a' = 0$, *and if* $a \neq 0$, *there exists* $a^* \in \mathcal{R}$ *such that* $aa^* = 1$.
5. $a(b+c)=ab+ac$

where $0 := [(0)]$ *and* $1 = [(1)]$.

Order properties

Proposition

The order on $\mathcal{R}$ is well defined.

Proposition

Let $\mathcal{R}^+$ be the positive elements of $\mathcal{R}$. Then, for $x \in \mathcal{R}$ either $x \in \mathcal{R}^+$, $x = 0$ or $-x \in \mathcal{R}^+$. Furthermore if $a, b \in \mathcal{R}^+$, then $a + b \in \mathcal{R}^+$ and $ab \in \mathcal{R}^+$.

Proposition (Order properties)

For all $a, b, c \in \mathcal{R}$, we have:

- i *Either $a < b$, $a = b$ or $b < a$.*
- ii *If $a < b$ and $b < c$, then $a < c$.*
- iii *If $a < b$, then $a + c < b + c$, and if $c > 0$, then $ac < bc$, whereas if $c < 0$, then $ac > bc$.*

Properties of $\mathcal{R}$

Proposition

The ordered field $\mathcal{R}$ contains $\mathbb{Q}$, and $\mathbb{Q}$ is dense in $\mathcal{R}$.

Proposition (Completeness of $\mathcal{R}$)

Every nonempty subset of $\mathcal{R}$ that is bounded above has a supremum.

Proposition (Uniqueness of $\mathcal{R}$)

If K is another complete ordered field, then there is a bijective map $f : K \to \mathcal{R}$ such that for $x, y \in K$, we have:

- i $f(x + y) = f(x) + f(y)$
- ii $f(xy) = f(x)f(y)$
- iii *If $x < y$, then $f(x) < f(y)$.*

In this case, we say that K is ***order isomorphic*** *to $\mathcal{R}$.*

Properties of infinite series

- Recall that $\sum_{k=1}^{\infty} a_k = A$ means that $\lim s_n = \lim(a_1 + \cdots a_n) = A$, so we can apply our results for sequences to get results for series.

Theorem (Algebraic limit theorem for series)

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then we have:

i $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$

ii $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Theorem (Cauchy criterion for series)

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > m \geq N$:

$$\left|a_{m+1} + a_{m+2} + \cdots + a_n\right| < \varepsilon$$

Corollary

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Properties of infinite series (cont.)

Theorem (Comparison test)

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

i. *If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
ii. *If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.*

- Remember that statements like this are true even if we change some finite number of initial terms. We can weaken the hypothesis to $0 \leq a_k \leq b_k$ for all $k \geq M$ for some $M \in \mathbb{N}$.

Theorem (Absolute convergence test)

If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges as well.

- Is the converse true?

Properties of infinite series (cont.)

Definition

If $\sum_{k=1}^{\infty} |a_k|$ converges, then we say the original series $\sum_{k=1}^{\infty} a_k$ **converges absolutely**. If on the other hand, the series $\sum_{k=1}^{\infty} a_k$ converges, but the series $\sum_{k=1}^{\infty} |a_k|$ diverges, then we say that the original series $\sum_{k=1}^{\infty} a_k$ **converges conditionally**.

Theorem (Alternating series test)

Let (a_n) be a sequence satisfying:

- i $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$
- ii $(a_n) \to 0$

Then, the alternating series $\sum_{k=1}^{\infty} (-1)^{n+1} a_n$ converges.

Example: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.