Lecture #8

MA 511, Introduction to Analysis

June 4, 2021

Limit points

Definition

A point x is a **limit point** of a set A if every ε-neighborhood $V_\varepsilon(x)$ of x intersects the set A at some point other than x. They are sometimes called "cluster points" or "accumulation points."

Theorem

A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

Definition

A point $a \in A$ is an **isolated point** of A if it is not a limit point of A.

- A set always contains its isolated points, but it may not contain its limit points.

Closed sets

Definition

A set $F \subseteq \mathbb{R}$ is **closed** if it contains its limit points.

Example: $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ and closed intervals $[a, b]$ are closed.

- There are sets which are both open and closed (sometimes called **clopen**) as well as sets that are neither open nor closed!

Theorem

A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.

Closures and complements

Definition

Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The **closure** of A is defined to be $\overline{A} = A \cup L$.

Theorem

For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.

Theorem

A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.

Theorem

i *The union of an finite collection of closed sets is closed.*

ii *The intersection of a arbitrary collection of closed sets is closed.*

Compact sets

Definition

A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

Example: The simplest example of a compact set is a closed interval $[c, d]$.

Definition

A set $A \subseteq \mathbb{R}$ is **bounded** if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem (Characterization of compactness in $\mathbb{R}$)

A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Theorem (Nested compact set property)

If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Open covers and the Heine-Borel theorem

Definition

Let $A \subseteq \mathbb{R}$. An **open cover** for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A. Given an open cover for A, a **finite subcover** is a finite subcollection of open sets from the original open cover whose union still contains A.

- Does the open cover $\left\{\left(\frac{1}{n}, 1\right) : n \in \mathbb{N}\right\}$ of $(0,1)$ have a finite subcover?

Theorem (Heine-Borel theorem)

Let K be a subset of $\mathbb{R}$. All of the following statements are equivalent in the sense that any one of them implies the other two:

i. *K is compact.*
ii. *K is closed and bounded.*
iii. *Every open cover for K has a finite subcover.*